

# A DIVISION ALGORITHM IN AN AFFINE FRAMEWORK FOR FLAT FAMILIES COVERING HILBERT SCHEMES

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**ABSTRACT.** We study the family of ideals  $\mathfrak{i} \subset R = K[x_1, \dots, x_n]$  whose quotients  $R/\mathfrak{i}$  share the same affine Hilbert polynomial and the same monomial  $K$ -vector basis, that we choose to be the sous-escalier  $\mathcal{N}(\mathfrak{j})$  of a strongly stable ideal  $\mathfrak{j} \subset R$ . The analogous problem for homogeneous ideals has already been studied, but in the non-homogeneous case there are more difficulties that we overcome introducing the notion of  $[\mathfrak{j}, m]$ -marked basis, for a fixed positive integer  $m$ . We design a division algorithm which works in an affine context and allows the explicit construction of a class of flat families of (non-homogeneous) ideals, that we call  $[\mathfrak{j}, m]$ -marked families. We can compute a set of equations endowing a marked family  $\mathcal{Mf}(\mathfrak{j}, m)$  with the structure of subscheme of a suitable affine space; moreover, we can simultaneously construct the homogenization of the ideals in  $\mathcal{Mf}(\mathfrak{j}, m)$  in a very efficient and simple way. Finally we show that, up to changes of coordinates, the marked families over strongly stable ideals in  $R$  give an open cover of Hilbert schemes. These results allow us to make explicit computations on Hilbert schemes, for example, for the one of 16 points in  $\mathbb{P}^7$ , we detect three irreducible components through a single point and we prove the smoothability of Gorenstein schemes with Hilbert function  $(1, 7, 7, 1)$ . In a similar way we also prove the smoothability of Gorenstein schemes with Hilbert function  $(1, 5, 5, 1)$ .

## 1. INTRODUCTION

In this paper, we are interested in computing the family of ideals  $\mathfrak{i} \subset R = K[x_1, \dots, x_n]$  whose quotients  $R/\mathfrak{i}$  share the same affine Hilbert polynomial and the same monomial  $K$ -vector basis, that we choose to be the sous-escalier  $\mathcal{N}(\mathfrak{j})$  of a strongly stable ideal  $\mathfrak{j} \subset R$ . A similar problem for homogeneous ideals has been studied in [12, 5, 6], but the non-homogeneous case is more tricky. In fact, in the homogeneous case the condition on the quotient implies the one on the Hilbert polynomial, while this is no more true for non-homogeneous ideals (see Example 4.5). Furthermore, for a fixed strongly stable ideal  $\mathfrak{j}$ , the family of ideals  $\mathfrak{i}$  such that  $\mathcal{N}(\mathfrak{j})$  is a basis for  $R/\mathfrak{i}$  may depend on an infinite number of parameters. For instance, for  $\mathfrak{j} = (x_2) \subset K[x_1, x_2]$  ( $x_2 > x_1$ ), the family of all ideals  $\mathfrak{i}$  such that  $K[x_1, x_2]/\mathfrak{i}$  is generated by  $\mathcal{N}(\mathfrak{j}) = \{x_1^n : n \in \mathbb{N}\}$  depends on infinitely many parameters.

We overcome these difficulties by the notion of  $[\mathfrak{j}, m]$ -marked basis, for a fixed positive integer  $m$  (see Definition 4.9). Similarly to the homogeneous case, any ideal generated by a  $[\mathfrak{j}, m]$ -marked basis satisfies both the above conditions and for  $m$  sufficiently high, the converse is true. In fact, for any ideal  $\mathfrak{i}$  generated by a  $[\mathfrak{j}, m]$ -marked basis, the quotient  $R/\mathfrak{i}$  has the same affine Hilbert function as  $R/\mathfrak{j}$  in degrees from  $m$  on. The collection of all the ideals generated by a  $[\mathfrak{j}, m]$ -marked basis is the  $[\mathfrak{j}, m]$ -marked family.

We face our study from both a theoretical and a computational point of view, obtaining two main results: a new division algorithm, that is able to detect  $[\mathfrak{j}, m]$ -marked bases, and the construction of a class of flat families of ideals.

Our division algorithm consists of steps of reduction based on the nice combinatorial structure of the strongly stable ideal  $\mathfrak{j}$ . For every ideal  $\mathfrak{i}$  generated by a  $[\mathfrak{j}, m]$ -marked basis, our

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algorithm is a new tool to compute normal forms modulo  $\mathfrak{i}$ . Moreover, it allows to explicitly compute equations defining the  $[\mathfrak{j}, m]$ -marked family.

A  $[\mathfrak{j}, m]$ -marked basis has many good properties of Gröbner bases, in a context where the ideal  $\mathfrak{j}$  covers a role analogous to that of an initial ideal and, at the same time, the strongly stable property addresses the lack of the term order. In particular, a  $[\mathfrak{j}, m]$ -marked basis behaves very well with respect to the homogenization.

Although we investigate  $[\mathfrak{j}, m]$ -marked families independently from the already studied homogeneous case, the new division algorithm refines and improves the so-called superminimal reduction, introduced in [5] for the homogeneous case. In fact, if  $J$  is a saturated strongly stable ideal in  $K[x_0, x_1, \dots, x_n]$ ,  $m$  is a positive integer and  $\mathfrak{j} = J \cap R$ , then any  $[\mathfrak{j}, m]$ -marked basis turns out to be the de-homogenization of a  $J_{\geq m}$ -marked basis introduced in [5].

On the other hand, the homogenization of a  $[\mathfrak{j}, m]$ -marked basis gives us the projective closure in  $\mathbb{P}^n$  of the affine scheme in  $\mathbb{A}^n$  defined by the ideal  $\mathfrak{i}$  it generates; moreover we can easily obtain a  $J_{\geq m}$ -marked basis from the  $[\mathfrak{j}, m]$ -marked basis. This good behavior is due to the fact that any projective scheme defined by a  $J_{\geq m}$ -marked basis does not contain components at infinity. These facts are also related to a strict correspondence between the satiety of the strongly stable ideal and that of the ideals in the homogeneous case.

The present "affine" division algorithm is computationally convenient in the general case since we skip time consuming multiplications by powers of  $x_0$ , unavoidable in the superminimal reduction described in [5]; moreover it is especially suited to treat the case of Artinian ideals.

We also show that every  $[\mathfrak{j}, m]$ -marked family is endowed with a structure of an affine scheme, in a very natural way, and is flat at  $\mathfrak{j}$ . For families of projective schemes, some criteria to recognize the flatness are available, when the Hilbert polynomial is fixed and the parameterizing scheme has nice properties. The affine case is more complicated: we apply the local characterization of flatness by the lifting of the syzygies of  $\mathfrak{j}$  to syzygies of the ideal generated by a  $[\mathfrak{j}, m]$ -marked basis. Our result is not obvious, because, in general, families of affine schemes can be non-flat, even if the affine Hilbert polynomial is fixed (see Example 5.9).

The flatness is the key feature of  $[\mathfrak{j}, m]$ -marked families allowing to embed them in suitable Hilbert schemes. The choice of the integer  $m$  is strategic to embed a marked family as a locally closed subscheme or as an open subset in a Hilbert scheme (see [6]).

Our computational and theoretical results allow to face open questions concerning the Hilbert scheme: in particular we show that every local Gorenstein Artin algebra with Hilbert function  $(1, 7, 7, 1)$  is smoothable.

The paper is organized in the following way. In Section 2, we recall some properties of strongly stable ideals and the main notions and results concerning the homogeneous case, i.e.  $J$ -marked families introduced in previous papers [12, 5, 6].

In Section 3, we focus our attention on relations among the regularity and satiety of  $J$  and those of the ideals of a  $J$ -marked family (Theorem 3.5).

In Section 4, we introduce the new notions used in the non-homogeneous context, in particular the new reduction relation (Definition 4.3) with its main features (Theorem 4.4), and develop the theory of  $[\mathfrak{j}, m]$ -marked bases and  $[\mathfrak{j}, m]$ -marked families (Subsection 4.1). We compare the new results with those of the homogeneous case (Subsection 4.2) and show the good behavior of  $[\mathfrak{j}, m]$ -marked sets, bases and the new described reduction relation with respect to the homogenization. Moreover, the new reduction relation produces a division algorithm that allows to construct  $[\mathfrak{j}, m]$ -marked families by an effective criterion (Subsection 4.3). In Subsection 4.4, a deep study of this criterion explains how the algorithm is particularly efficient in the Artinian case (Theorem 4.30). This fact is also related to an *optimal* choice for  $m$  suggested by Theorem 4.27.

In Section 5, we describe the structure of affine scheme of a  $[j, m]$ -marked family (Theorem 5.3) and study its flatness, giving also a particular example of non-flat affine family with fixed Hilbert polynomial (Example 5.9). Furthermore, we show that the elements of a  $[j, m]$ -marked family can be simultaneously homogenized (Proposition 5.10) and that we can obtain an open cover of  $\text{Hilb}_{p(t)}^n$ , the Hilbert scheme parameterizing subschemes of  $\mathbb{P}^n$  which share the same Hilbert polynomial with  $S/j^h$  (Theorem 5.11).

In Section 6 we describe an algorithm which computes a set of equations defining a  $[j, m]$ -marked family, by our reduction process.

In Section 7, as an application of our results, we analyze the  $[j, 3]$ -marked family for a suitable strongly stable ideal  $j \subset R = K[x_1, \dots, x_7]$  such that  $R/j$  has affine Hilbert polynomial 16. We show that  $\text{Mf}(j, 3)$  has at least three irreducible components; one of them contains the reduced schemes of length 16 and we prove in particular that it contains the Gorenstein schemes with Hilbert function  $(1, 7, 7, 1)$ . In a similar way we also prove the smoothability of Gorenstein schemes with Hilbert function  $(1, 5, 5, 1)$ . This last result has been independently obtained by J. Jelisiejew in [24] by different tools.

## 2. NOTATIONS AND BACKGROUND

Let  $K$  be a field. We consider the polynomial rings  $S = K[x_0, \dots, x_n] \supset R = K[x_1, \dots, x_n]$ , with variables ordered as  $x_n > \dots > x_1 > x_0$ . In this section, we fix some notations and facts that hold both in  $S$  and in  $R$ . When needed, we will state precisely in which ring we are working to avoid any ambiguity.

We will denote by capital letters, such as  $I$  and  $J$ , homogeneous ideals in  $S$ , while we will use gothic letters, such as  $\mathfrak{i}$  and  $\mathfrak{j}$ , for ideals (not necessarily homogeneous) in  $R$ .

For any set  $A$  of polynomials, we will denote by  $A_t$  the subset of  $A$  made up of homogeneous polynomials of degree  $t$  and by  $A_{\leq t}$  the set of elements of  $A$  of total degree  $\leq t$ .

For a homogeneous ideal  $I$  of  $S$ , the *Hilbert function* of  $S/I$  is the function  $H_{S/I} : \ell \in \mathbb{N} \rightarrow \dim_K \frac{S_\ell}{I_\ell} \in \mathbb{N}$  and the *Hilbert polynomial*  $P_{S/I}(t)$  is the unique numerical polynomial in  $\mathbb{Q}[t]$  such that  $P_{S/I}(\ell) = H_{S/I}(\ell)$ , for  $\ell \gg 0$ . For an ideal  $\mathfrak{i}$  of  $R$ , the *affine Hilbert function* of  $R/\mathfrak{i}$  is the function  ${}^aH_{R/\mathfrak{i}} : \ell \in \mathbb{N} \rightarrow \dim_K \frac{R_{\leq \ell}}{\mathfrak{i}_{\leq \ell}} \in \mathbb{N}$  and the *affine Hilbert polynomial*  ${}^aP_{R/\mathfrak{i}}(t)$  is the unique numerical polynomial in  $\mathbb{Q}[t]$  such that  ${}^aP_{R/\mathfrak{i}}(\ell) = {}^aH_{R/\mathfrak{i}}(\ell)$ , for  $\ell \gg 0$ . We refer to ([13, Chapter 9, Section 3], [25, Section 5.6]) for basic results about these functions.

Given  $F \in S$ , we denote by  $F^a \in R$  the polynomial obtained by replacing  $x_0$  with 1 in  $F$ . Conversely, if  $\mathfrak{f} \in R$ , then we denote by  $\mathfrak{f}^h$  the polynomial  $x_0^{\deg \mathfrak{f}} \mathfrak{f}(x_1/x_0, \dots, x_n/x_0) \in S$ .

**2.1. Strongly stable ideals.** If  $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$  belongs to  $\mathbb{N}^{n+1}$ , we use the compact notation  $x^\alpha$  to represent the monomial  $x_0^{\alpha_0} x_1^{\alpha_1} \dots x_n^{\alpha_n} \in S$ . We adopt the analogous notation for monomials in  $R$ .

We denote by  $\max(x^\alpha)$  the biggest variable that appears in  $x^\alpha$  and, analogously,  $\min(x^\alpha)$  is the smallest variable that appears in  $x^\alpha$ . For a monomial ideal  $J \subset S$  (resp.  $\mathfrak{j} \subset R$ ), we denote by  $B_J$  (resp.  $B_{\mathfrak{j}}$ ) its *monomial basis* and by  $\mathcal{N}(J)$  (resp.  $\mathcal{N}(\mathfrak{j})$ ) its *sous-escalier*, that is the set of monomials in  $S \setminus J$  (resp.  $R \setminus \mathfrak{j}$ ).

We will say that a monomial  $x^\beta$  can be obtained by a monomial  $x^\alpha$  through an *elementary move* if  $x^\alpha x_j = x^\beta x_i$  for some variable  $x_i \neq x_j$ . In particular, if  $i < j$ , we say that  $x^\beta$  can be obtained by  $x^\alpha$  through an *increasing* elementary move and we write  $x^\beta = e_{i,j}^+(x^\alpha)$ , whereas if  $i > j$  the move is said to be *decreasing* and we write  $x^\beta = e_{i,j}^-(x^\alpha)$ . The transitive closure of the relation  $x^\beta > x^\alpha$  if  $x^\beta = e_{i,j}^+(x^\alpha)$  gives a partial order on the set of monomials of a fixed degree, that we will denote by  $>_B$  and that is often called *Borel partial order*:

$$x^\beta >_B x^\alpha \iff \exists x^{\gamma_1}, \dots, x^{\gamma_t} \text{ such that } x^{\gamma_1} = e_{i_0, j_0}^+(x^\alpha), \dots, x^\beta = e_{i_t, j_t}^+(x^{\gamma_t})$$

for suitable indexes  $i_k, j_k$ . In analogous way, we can define the same relation using decreasing moves:

$$x^\beta >_B x^\alpha \iff \exists x^{\delta_1}, \dots, x^{\delta_s} \text{ such that } x^{\delta_1} = e_{h_0, l_0}^-(x^\beta), \dots, x^\alpha = e_{h_s, l_s}^-(x^{\delta_s})$$

for suitable indexes  $i_k, j_k$ .

Note that every term order  $\succ$  is a refinement of the Borel partial order  $>_B$ , that is  $x^\alpha >_B x^\beta$  implies that  $x^\alpha \succ x^\beta$ .

**Definition 2.1.** A monomial ideal  $J \subset S$  (or  $\mathfrak{j} \subset R$ ) is said to be *strongly stable* if every monomial  $x^\alpha$  such that  $x^\alpha >_B x^\beta$ , with  $x^\beta \in J$  (resp.  $x^\beta \in \mathfrak{j}$ ), belongs to  $J$  (resp. to  $\mathfrak{j}$ ).

Both in  $S$  and  $R$ , a strongly stable ideal is always Borel fixed, that is fixed by the action of the Borel subgroup of upper triangular matrices of  $GL(n+1)$  for  $S$  and of  $GL(n)$  for  $R$ . The vice versa holds under the hypothesis  $\text{char}(K) = 0$  (e.g. [15]). Throughout the paper we will work on the field  $K$  without any further hypothesis on its characteristic. We neither need to assume that  $K$  is algebraically closed: indeed, even when we recall the application of the techniques we develop to the study of the Hilbert Scheme, we consider the classical construction of [23, Appendix C], [10], recalled also in [6].

Galligo's Theorem [18], generalized in [4] for a field of non-zero characteristic, guarantees that in generic coordinates the initial ideal of  $I$  (homogeneous or not), w.r.t. a fixed term order, is a constant Borel fixed monomial ideal called the *generic initial ideal* of  $I$ .

The following statements and definitions concern strongly stable ideals both in  $S$  and  $R$ .

**Definition 2.2.** Given a strongly stable ideal  $J$ , with monomial basis  $B_J$ , and a monomial  $x^\gamma \in J$ , we define

$$x^\gamma = x^\alpha *_J x^\eta \text{ with } \gamma = \alpha + \eta, \quad x^\alpha \in B_J \text{ and } \min(x^\alpha) \geq \max(x^\eta).$$

This decomposition exists and it is unique (see [16, Lemma 1.1]).

**Lemma 2.3.** Let  $J$  be a strongly stable ideal. If  $x^\epsilon$  belongs to  $\mathcal{N}(J)$  and  $x^\epsilon \cdot x^\delta = x^{\epsilon+\delta}$  belongs to  $J$  for some  $x^\delta$ , then  $x^{\epsilon+\delta} = x^\alpha *_J x^\eta$  with  $x^\eta <_{\text{Lex}} x^\delta$ . Furthermore if  $|\delta| = |\eta|$ , then  $x^\eta <_B x^\delta$ .

*Proof.* See [5, Lemma 2.4]. □

**Definition 2.4.** Let  $J$  be a strongly stable ideal. On the monomials of  $J$  we define the following total order:

$$x^\alpha *_J x^\delta <_* x^{\alpha'} *_J x^{\delta'} \text{ if } x^\delta <_{\text{Lex}} x^{\delta'} \text{ or } x^\delta = x^{\delta'} \text{ and } x^\alpha <_{\text{Lex}} x^{\alpha'}.$$

In Definition 2.4, in order to obtain a well-order on the monomials of  $J$ , it is sufficient to order the monomials in  $B_J$  in any way, not necessarily lexicographically. We underline that  $<_*$  is a well-order on  $J$ , but it is not a term order, as shown by the following example

**Example 2.5.** Consider  $J = (x_3^2, x_3x_2, x_3x_1, x_2^2, x_2x_1)$  in  $K[x_0, x_1, x_2, x_3]$ , with  $x_0 < x_1 < x_2 < x_3$ . If we take  $x_0x_2x_3$  and  $x_1^2x_2$ , we get  $x_2x_3 *_J x_0 <_* x_1x_2 *_J x_1$ , but  $x_1^2x_2x_3 = x_3x_2 *_J x_1^2 <_* x_3^2 *_J x_2x_0 = x_3^2x_2x_0$ .

**2.2. Marked sets and bases for homogeneous ideals.** For a polynomial  $F$ , belonging to  $S$  or  $R$ , we denote by  $\text{supp}(F)$  its *support*, that is the set of monomials appearing in  $F$  with non-zero coefficients.

**Definition 2.6.** [31, 12] A *marked polynomial* is a polynomial  $F \in S$  together with a specified monomial of  $\text{supp}(F)$  that will be called *head term* of  $F$  and denoted by  $\text{Ht}(F)$ .

Let  $J$  be a monomial ideal in  $S$ . A finite set  $G \subset S$  of homogeneous marked polynomials  $F_\alpha$  is a *J-marked set* if the head terms  $\text{Ht}(F_\alpha) = x^\alpha$  are pairwise different, they form the

monomial basis of  $J$  and the monomials in  $\text{supp}(x^\alpha - F_\alpha)$  do not belong to  $J$ , i.e.  $|\text{supp}(F_\alpha) \cap J| = 1$ . We call *tail of  $F_\alpha$*  the polynomial  $T(F_\alpha) = \text{Ht}(F_\alpha) - F_\alpha$ , so we can write  $F_\alpha = x^\alpha - T(F_\alpha)$ .

A  $J$ -marked set  $G \subseteq S$  is a  $J$ -marked basis if  $(G) \oplus \mathcal{N}(J) = S$  as a  $K$ -vector space.

**Definition 2.7.** [31, 12] The family of all homogeneous ideals  $I$  such that  $\mathcal{N}(J)$  is a basis of the quotient  $S/I$  as a  $K$ -vector space will be denoted by  $\mathcal{Mf}(J)$  and called  $J$ -marked family.

Referring to [12, 5], we recall some properties and basic results about  $J$ -marked bases and superminimals generators that will be useful in the next sections.

For a  $J$ -marked set  $G$  with  $J$  strongly stable and for every integer  $\ell \geq \min\{t : J_t \neq (0)\}$ , we set

$$V_\ell := \{x^\delta f_\alpha \mid f_\alpha \in G, |\delta + \alpha| = \ell \text{ and } x^\delta x^\alpha = x^\alpha *_J x^\delta\} \text{ and } V := \cup_\ell V_\ell.$$

The ideal  $I$  generated by  $G$  belongs to  $\mathcal{Mf}(J)$  if and only if  $I_t = \langle V_t \rangle$  for every  $t$ , by [5, Lemma 2.2].

In [12, 5] the reduction relation  $\xrightarrow{V_\ell}$  on homogeneous polynomials of degree  $\ell$ , in the usual sense of Gröbner bases theory, is introduced and studied to investigate properties and features of  $\mathcal{Mf}(J)$ . An experimental version of the algorithms for  $J$ -marked bases which use the reduction relation  $\xrightarrow{V_\ell}$  is available in [9] for the software package Singular [14].

Anyway, to study the family  $\mathcal{Mf}(J_{\geq m})$  for a saturated strongly stable ideal  $J$  and a given integer  $m$ , a much more efficient reduction relation can be used.

To recall the definition of this latter reduction relation, called *superminimal reduction*, first we need the so-called *superminimal generators of  $J_{\geq m}$* , i.e. the monomials of  $B_{J_{\geq m}}$  of type  $x_0^{t_\alpha} x^\alpha$ , with  $x^\alpha$  in  $B_J$  and  $t_\alpha \geq 0$ . We denote the set of superminimal generators of  $J_{\geq m}$  by  $sB_{J_{\geq m}}$ . Then we define the subset  $sG$  of the polynomials of a  $(J_{\geq m})$ -marked set  $G$  whose head terms are the monomials in  $sB_{J_{\geq m}}$ . The subset  $sG$  is called a  $(J_{\geq m})$ -marked *superminimal set* and, when  $G$  is a  $(J_{\geq m})$ -marked basis,  $sG$  is called a  $(J_{\geq m})$ -*superminimal basis* [5, Definitions 3.5 and 3.9].

**Definition 2.8.** [5, Definition 3.11] Consider a saturated strongly stable ideal  $J$ , a positive integer  $m$ , a  $J_{\geq m}$ -marked set  $G$  and two polynomials  $g$  and  $g_1$ . We say that  $g$  is in  $sG_*$ -relation with  $g_1$  if there is a monomial  $x^\gamma \in \text{supp}(g) \cap J_{\geq m}$ , such that  $x^\gamma$  is divisible by a superminimal generator  $x^{\alpha'} = x_0^{t_\alpha} x^\alpha$  of  $sB_{J_{\geq m}}$ , with  $x^\gamma = x^\alpha *_J x^{\eta'} = x_0^{t_\alpha} x^\alpha \cdot x^{\eta'}$  and  $g_1 = g - c_\gamma \cdot x^{\eta'} f_{\alpha'}$  (where  $c_\gamma$  is the coefficient of  $x^\gamma$  in  $g$ ); hence,  $g_1$  is obtained by replacing the monomial  $x^\gamma$  in  $g$  by  $x^{\eta'} \cdot T(f_{\alpha'})$ . We call *superminimal reduction* the transitive closure of the above relation and denote it by  $\xrightarrow{sG_*}$ . A homogeneous polynomial  $h$  is *strongly reduced* if no monomial in  $\text{supp}(h)$  is divisible by a monomial of  $B_J$ .

**Theorem 2.9.** [5, Theorem 3.14] Let  $J$  be a saturated strongly stable ideal in  $S$  and  $sG$  be a  $J_{\geq m}$ -marked superminimal set. Then:

- (i)  $\xrightarrow{sG_*}$  is Noetherian.
- (ii) For every homogeneous polynomial  $g$  there exist  $t$  and a unique polynomial  $g(t)$  strongly reduced such that  $x_0^t \cdot g \xrightarrow{sG_*} g(t)$ . If  $\bar{t}$  is the minimum one and  $\bar{g} := g(\bar{t})$ , then  $g(t) = x_0^{t-\bar{t}} \cdot \bar{g}$  for every  $t \geq \bar{t}$ . There is an effective procedure that computes  $\bar{t}$  and  $\bar{g}$ .

If moreover  $sG$  is the superminimal basis of an ideal  $I$  of  $\mathcal{Mf}(J_{\geq m})$ , then  $\xrightarrow{sG_*}$  computes the  $J_{\geq m}$ -normal forms modulo  $I$ . More precisely, for every homogeneous polynomial  $g$ :

$$\text{Nf}(g) = \begin{cases} g, & \text{if } \deg(g) < m \\ \bar{g}/x_0^{\bar{t}}, & \text{if } \deg(g) \geq m \text{ and } x_0^{\bar{t}} \cdot g \xrightarrow{sG_*} \bar{g} \end{cases}$$



We refer to [5] for other properties of the superminimal reduction. Here, we recall that this reduction is not only very efficient for effective computations, but it is also a very good tool for proving some feature of the family  $\mathcal{Mf}(J_{\geq m})$ . For example, the superminimal reduction is the tool used in [5] to prove that there is a scheme-theoretical isomorphism between  $\mathcal{Mf}(J_{\geq m})$  and  $\mathcal{Mf}(J_{\geq m-1})$ , for every  $m$  higher than or equal to a suitable integer  $\rho$  which depends only on  $J$  [5, Proposition 5.7] and which will be studied in Subsection 4.2.

Even though this procedure allows computations on Hilbert schemes (see [6]) which we cannot cope with by using Gröbner techniques, it is plain that the multiplication of  $x_0^t$  with every monomial of the polynomial to reduce by  $\xrightarrow{sG_*}$  is time-consuming in any implementation of the Algorithm presented in [5].

### 3. REGULARITY AND SATIETY IN A $J$ -MARKED FAMILY

**Definition 3.1.** Let  $I$  be a homogeneous ideal in  $S$ . Consider its graded minimal free resolution

$$0 \rightarrow E_n \rightarrow \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow I \rightarrow 0,$$

where  $E_i = \oplus_j S(-a_{ij})$ . The ideal  $I$  is  $m$ -regular if  $m \geq a_{ij} - i$  for every  $i, j$ . The *regularity* of  $I$ , denoted by  $\text{reg}(I)$ , is the smallest  $m$  for which  $I$  is  $m$ -regular.

The ideal  $I$  is *saturated* if  $I = (I : (x_0, \dots, x_n))$ . The *saturation* of  $I$  is  $I^{\text{sat}} = \cup_{j \geq 0} (I : (x_0, \dots, x_n)^j)$  and  $I$  is  $m$ -saturated if  $I_t = I_t^{\text{sat}}$  for every  $t \geq m$ . The *satiety* of  $I$ , denoted by  $\text{sat}(I)$ , is the smallest  $m$  for which  $I$  is  $m$ -saturated.

A homogeneous polynomial  $h$  in  $S$  is *generic* for  $I$  if  $h$  is a non-zero divisor in  $S/(I^{\text{sat}})$ .

Strongly stable ideals have a very nice combinatorial structure which allows us to easily read their *regularity* and *satiety* from their monomial bases.

**Lemma 3.2.** Let  $J$  be a strongly stable ideal in  $S$  with monomial basis  $B_J$ . Then:

$$\text{reg}(J) = \max\{\deg(x^\alpha) : x^\alpha \in B_J\},$$

$$\text{sat}(J) = \max\{\deg(x^\alpha) : x^\alpha \in B_J \text{ and } x_0 | x^\alpha\}.$$

Furthermore  $(J : x_0^\infty) = J^{\text{sat}}$ . As a consequence, a strongly stable ideal is saturated if and only if no monomial in its monomial basis involves the smallest variable of the ring.

*Proof.* See [3, Proposition 2.9] and [20, Proposition 2.9, Corollary 2.10].  $\square$

Recall that, if  $\text{in}_<(I)$  is the ideal of the leading terms of an ideal  $I$  with respect to some term order  $<$ , then  $\text{reg}(I) \leq \text{reg}(\text{in}_<(I))$ . In particular, when  $\text{char}(K) = 0$ , if  $\text{gin}(I)$  is the generic initial ideal of  $I$  with respect to  $\text{DegRevLex}$ , then  $\text{reg}(I) = \text{reg}(\text{gin}(I))$  and  $\text{sat}(I) = \text{sat}(\text{gin}(I))$  [3, 4]. For what concerns the regularity, we highlight that the ideal  $J$  plays for  $I \in \mathcal{Mf}(J)$  a role analogous to that of an initial ideal. Indeed, we recall the following result.

**Proposition 3.3.** [12, Proposition 4.6] Let  $J$  be a strongly stable ideal in  $S$ . The  $J$ -marked family  $\mathcal{Mf}(J)$  is flat at the origin. In particular, for every ideal  $I \in \mathcal{Mf}(J)$ , we get  $\text{reg}(J) \geq \text{reg}(I)$ .

For what concerns satiety and saturation, the strongly stable ideal  $J$  plays for  $I \in \mathcal{Mf}(J)$  a similar role to the one of the generic initial ideal with respect to  $\text{DegRevLex}$  in the following sense. First, observe that we can read the second part of the statement of Lemma 3.2 in this geometric way: the associated scheme  $\mathcal{Z} = \text{Proj}(S/J^{\text{sat}})$  in the projective space  $\mathbb{P}^n$  has no components lying in the hyperplane  $H_0 \subseteq \mathbb{P}^n$  defined by the ideal  $(x_0)$ , because  $J^{\text{sat}}$  does not have any primary component contained in the ideal  $(x_0)$ . As a consequence, the affine scheme  $\mathcal{Z} \setminus H_0$  shares some numerical features with  $\mathcal{Z} \subseteq \mathbb{P}^n$ , as the affine Hilbert function and polynomial.

We now show that the same holds for every homogeneous ideal in  $S$  generated by a  $J$ -marked basis, with  $J$  a strongly stable ideal; not only the satiety of  $J$  bounds the satiety of every ideal in  $\mathcal{Mf}(J)$ , but also, for every  $I \in \mathcal{Mf}(J)$ , the scheme  $\text{Proj}(S/I)$  has no components at infinity.

We need the following Lemma.

**Lemma 3.4.** *Let  $J$  be a strongly stable ideal in  $S$ ,  $m$  be an integer,  $m \geq \text{sat}(J)$ ,  $g$  a homogeneous polynomial of degree  $\ell \geq m$  and  $G$  a  $J$ -marked set. Then*

$$g \in \langle V_\ell \rangle \Leftrightarrow x_0 \cdot g \in \langle V_{\ell+1} \rangle.$$

*Proof.* The proof is the same as the one in [5, Lemma 4.2], we simply need to observe that for every monomial  $x_0 x^\gamma \in J$ , with  $x^\gamma \in \text{supp}(g)$ , we obtain  $x_0 x^\gamma \notin B_J$  because  $m \geq \text{sat}(J)$ .  $\square$

**Theorem 3.5.** *Let  $J$  be a strongly stable ideal in  $S$  and  $I$  be an ideal belonging to  $\mathcal{Mf}(J)$ . Then:*

- (i)  $x_0$  is generic for  $I$ ;
- (ii) if  $J$  is saturated, then  $I$  is saturated;
- (iii)  $\text{sat}(I) \leq \text{sat}(J)$ .

As a consequence,  $(I : x_0^\infty) = I^{\text{sat}}$ .

*Proof.* Recall that  $I = \langle V \rangle$ , because  $I$  belongs to  $\mathcal{Mf}(J)$  [5, Lemma 2.2]. Hence, assuming that  $J$  is  $m$ -saturated, by Lemma 3.4 we get that for every  $t \geq m$

$$f \in (I : x_0)_t \Rightarrow f x_0 \in I_{t+1} = \langle V \rangle_{t+1} \Rightarrow f \in \langle V \rangle_t = I_t,$$

or in other words

$$(I : x_0)_t = I_t, \text{ for every } t \geq m.$$

By [3, Lemma (1.6)], this is equivalent to the fact that  $I$  is  $m$ -saturated too and  $x_0$  is generic for  $I$ .  $\square$

**Remark 3.6.** If  $K$  is not an infinite field, there may not be a general linear form for an ideal  $I$ . However, we do not need this hypothesis on  $K$ : even if  $K$  is finite,  $x_0$  is a general linear form for the strongly stable ideal  $J$ , and by Theorem 3.5, it is general for  $I \in \mathcal{Mf}(J)$  too.

**Corollary 3.7.** *Let  $J$  be a non-null saturated strongly stable ideal in  $S$  and  $m$  be a positive integer. If  $I$  is a non-saturated ideal of  $\mathcal{Mf}(J_{\geq m})$ , then  $\text{sat}(I) = m$  and  $I = (I^{\text{sat}})_{\geq m}$ .*

*Proof.* Since  $I$  is not saturated, then by Theorem 3.5 the ideal  $J_{\geq m}$  is not saturated, but  $\text{sat}(J) = m$ . Since  $m$  is also the initial degree of the ideal  $I$ , we also have that  $\text{sat}(I) \geq m$ . By Theorem 3.5(iii), we have  $\text{sat}(I) = m$ . In particular, we get  $I = (I^{\text{sat}})_{\geq m}$ .  $\square$

The family  $\mathcal{Mf}(J)$ , with  $J$  a strongly stable ideal, in general contains also saturated ideals, even if  $J$  is not saturated, as shown in the following examples.

**Example 3.8.** Take the saturated strongly stable ideal  $J = (x_2^2, x_1^3 x_2, x_1^4) \subseteq S = K[x_0, x_1, x_2]$ , for which  $S/J$  has Hilbert polynomial 7. Geometrically,  $\text{Proj}(S/J)$  is a non-reduced scheme of length 7. The family  $\mathcal{Mf}(J_{\geq 3})$  is a dense open subset of  $\mathcal{Hilb}_7^2$  [6], which has only one component. Since 7 general points in  $\mathbb{P}^2$  are not on a conic, there is an open subset of  $\mathcal{Mf}(J)$  made up of saturated ideals. More explicitly, we consider the  $J_{\geq 3}$ -marked set:

$$G = B_J \setminus \{x_2^2 x_0, x_1^4\} \cup \{x_0 x_2^2 - u x_1^2 x_2, x_1^4 + v u x_1^2 x_0^2 - v x_2 x_0^3\}, \quad u, v \in K$$

and define  $I := (G)K[x_0, x_1, x_2]$ . Since  $\dim I_t = \dim J_t$  for every  $t \geq 3$ ,  $I \in \mathcal{Mf}(J_{\geq 3})$  for every  $u, v \in K$  and  $I$  is saturated.

**Example 3.9.** Let  $I$  be the saturated ideal in  $K[x_0, \dots, x_3]$  defining the reduced scheme made up of the following five points in  $\mathbb{P}^3$ :

$$[1 : 4 : 3 : 0], \quad [2 : 0 : 1 : 1], \quad [3 : 7 : 5 : 6], \quad [1 : 1 : 0 : 1], \quad [0 : 1 : 0 : 1].$$

We consider the generic initial ideal of  $I$  with respect to **DegLex**:

$$J := \text{gin}_{\text{DegLex}}(I) = (x_0x_1x_2, x_0^3x_2, x_1^5, x_3^2, x_2x_3, x_1x_3, x_0x_3, x_2^2, x_1^2x_2).$$

$J$  is a strongly stable ideal with satiety 4 and  $I$  belongs to  $\mathcal{Mf}(J)$ , because its  $J$ -marked basis is exactly its reduced Gröbner basis with respect to **DegLex**.

#### 4. DEHOMOGENIZING MARKED SETS AND BASES

By Theorem 3.5, we can think that any kind of computations performed on a homogeneous ideal  $I$  generated by a  $J$ -marked basis could be performed after a process of “affinization”. In the present section, we will see in which sense this is true, showing that the superminimal reduction of Definition 2.8 has a natural correspondence with a reduction process for the non-homogeneous case.

The starting point to establish a correspondence between the homogeneous and the non-homogeneous case is very simple: if  $J$  is a saturated strongly stable ideal in  $S$ , then  $J \cap R = (B_J)R$  is a strongly stable ideal too. Viceversa, if  $j$  is a strongly stable ideal in  $R$ , then  $(B_j)S$  is a saturated strongly stable ideal in  $S$ .

We are going to define a reduction process for non-homogeneous polynomials that will lead to interesting computational and theoretical results: on the one hand, this process will give a deeper insight in the theory of marked bases over a strongly stable ideal in the homogeneous case, for what concerns 0-dimensional ideals (Corollary 4.30); on the other hand, the notion of non-homogeneous marked basis (Definition 4.9) will lead to the construction of a flat family of non-homogeneous ideals (Section 5).

We will describe the relations between the homogeneous case and the non-homogeneous one, observing how to pass from one to the other. In particular, by the homogenization of the non-homogeneous marked bases we can obtain a  $J$ -marked family from a non-homogeneous one and vice versa.

**4.1.  $[j, m]$ -marked sets, completions and bases in  $R$ .** The polynomial  $f \in R$  is a marked polynomial of  $R$  if it is marked as a polynomial in  $S$  (according to the first part of Definition 2.6).

**Definition 4.1.** Let  $j$  be a strongly stable ideal in  $R$ . We call *non-homogeneous  $j$ -marked set* (*n.h.  $j$ -marked set*, for short) a finite subset  $\mathfrak{G} \subset R$  such that every polynomial  $f_\alpha \in \mathfrak{G}$  is marked and the set  $\{\text{Ht}(f_\alpha)\}$  is the monomial basis of  $B_j$ . We call *tail* of  $f_\alpha$  the polynomial  $T(f_\alpha) = \text{Ht}(f_\alpha) - f_\alpha$ .

**Remark 4.2.** The polynomials in a n.h.  $j$ -marked set are not supposed to be homogeneous, while the polynomials in Definition 2.6 are.

**Definition 4.3.** Let  $j$  be a strongly stable ideal in  $R$  and let  $\mathfrak{G} \subset R$  be a n.h.  $j$ -marked set. We say that the polynomial  $\mathfrak{g} \in R$  is in  $\mathfrak{G}_*$ -relation with  $\mathfrak{g}_1 \in R$  if there is a monomial  $x^\gamma \in \text{supp}(\mathfrak{g}) \cap j$ ,  $x^\gamma = x^\alpha * x^\eta$ , and  $\mathfrak{g}_1 = \mathfrak{g}_\gamma - c_\gamma \cdot x^\eta f_\alpha$ , where  $c_\gamma$  is the coefficient of  $x^\gamma$  in  $\mathfrak{g}$ . In other words,  $\mathfrak{g}_1$  is obtained by replacing in  $\mathfrak{g}$  the monomial  $x^\gamma$  by  $x^\eta \cdot T(f_\alpha)$ . We call  $\mathfrak{G}_*$ -reduction the transitive closure of the above relation and denote it by  $\xrightarrow{\mathfrak{G}_*}$ . Moreover, we say that:

- $\mathfrak{g}$  can be reduced to  $\mathfrak{g}_1$  by  $\xrightarrow{\mathfrak{G}_*}$  if  $\mathfrak{g} \xrightarrow{\mathfrak{G}_*} \mathfrak{g}_1$  ;
- $\mathfrak{g}$  is reduced if  $\text{supp}(\mathfrak{g}) \cap j = \emptyset$ , i.e.  $\mathfrak{g}$  is not further reducible by  $\xrightarrow{\mathfrak{G}_*}$ .



The following result states that every polynomial  $\mathfrak{g}$  of  $R$  is in  $\mathfrak{G}_*$ -relation with a reduced polynomial. Hence, we have a division algorithm to construct reduced polynomials, also in a context in which we deal with non-homogeneous polynomials.

**Theorem 4.4.** *Given a n.h.  $\mathfrak{j}$ -marked set  $\mathfrak{G}$ , the  $\mathfrak{G}_*$ -reduction  $\xrightarrow{\mathfrak{G}_*}$  is Noetherian and for every polynomial  $\mathfrak{g} \in R$  there is a reduced polynomial  $\bar{\mathfrak{g}}$  such that  $\mathfrak{g} \xrightarrow{\mathfrak{G}_*} \bar{\mathfrak{g}}$ .*

*Proof.* It is sufficient to prove both statements for monomials. If  $x^\beta \notin \mathfrak{j}$  then it is reduced and there is nothing to prove. We then consider monomials in  $\mathfrak{j}$  and proceed by induction on  $<_*$ .

For every  $x^\alpha \in B_{\mathfrak{j}}$ , we reduce  $x^\alpha$  by  $\mathfrak{f}_\alpha$ :  $x^\alpha \xrightarrow{\mathfrak{G}_*} T(\mathfrak{f}_\alpha)$ , which is a reduced polynomial.

We now consider  $x^\beta = x^\alpha * x^\delta \in \mathfrak{j} \setminus B_{\mathfrak{j}}$  and assume that the thesis holds for every monomial in  $\mathfrak{j}$  which is smaller than  $x^\alpha * x^\delta$  with respect to  $<_*$ . We perform the first step of reduction on  $x^\beta = x^\alpha *_{\mathfrak{j}} x^\delta$  using  $\mathfrak{f}_\alpha$ :  $x^\beta = x^\alpha x^\delta \xrightarrow{\mathfrak{G}_*} x^\delta T(\mathfrak{f}_\alpha)$ . If  $\text{supp}(x^\delta T(\mathfrak{f}_\alpha)) \subseteq \mathcal{N}(\mathfrak{j})$ , we are done. Otherwise, for every  $x^\gamma \in \text{supp}(x^\delta T(\mathfrak{f}_\alpha)) \cap \mathfrak{j}$ , we have  $x^\gamma = x^{\alpha'} *_{\mathfrak{j}} x^{\delta'} <_* x^\alpha x^\delta = x^\beta$  by Lemma 2.3. Then, by the inductive hypothesis, applying a finite number of steps of  $\xrightarrow{\mathfrak{G}_*}$  on  $x^\gamma$ , we get a reduced polynomial.  $\square$

Given a strongly stable ideal  $\mathfrak{j}$ , then the notions of n.h.  $\mathfrak{j}$ -marked set  $\mathfrak{G}$  and the related  $\mathfrak{G}_*$ -reduction  $\xrightarrow{\mathfrak{G}_*}$  seem to be the right tool to read some of the numerical invariants of the ideal generated by  $\mathfrak{G}$  in  $R$ , in particular the affine Hilbert function. However, without any further assumption on the polynomials of  $\mathfrak{G}$ ,  $\mathfrak{j}$  and  $(\mathfrak{G})$  do not share the same affine Hilbert polynomial. Further, the following simple example shows that in our definition of  $\mathfrak{j}$ -marked set, we do not have a bound on the degree of the reduced polynomials we obtain by  $\xrightarrow{\mathfrak{G}_*}$ .

**Example 4.5.** Consider  $\mathfrak{j} = (x_3, x_2^2) \subseteq K[x_1, x_2, x_3]$  and consider a n.h.  $\mathfrak{j}$ -marked set of polynomials  $\mathfrak{G} = \{f_1 = x_3 - x_1^4, f_2 = x_2^2\} \subset R$ . The quotient  $R/(\mathfrak{G})$  has affine Hilbert polynomial  $8t - 8$ , while  $R/\mathfrak{j}$  has affine Hilbert polynomial  $2t + 1$ .

The monomial  $x_3^\ell = x_3 *_{\mathfrak{j}} x_3^{\ell-1}$  belongs to  $\mathfrak{j}$  and is reduced by  $\xrightarrow{\mathfrak{G}_*}$  to  $\mathfrak{g} := x_3^{\ell-1} T(f_1) = x_3^{\ell-1} x_1^4$ . Of course  $\mathfrak{g}$  is not yet reduced because its support is contained in  $\mathfrak{j}$ . With further steps of reduction, we obtain  $x_1^{4\ell}$  which is not further reducible.

We introduce the following more refined notion to have a control on the degree of the reduced polynomials we obtain by  $\xrightarrow{\mathfrak{G}_*}$ .

**Definition 4.6.** Let  $\mathfrak{j}$  be a strongly stable ideal in  $R$  and  $m$  be a positive integer. A n.h.  $\mathfrak{j}$ -marked set  $\mathfrak{G}$  is a  $[\mathfrak{j}, m]$ -marked set if for every  $\mathfrak{f}_\alpha \in \mathfrak{G}$ , we have  $\text{supp}(T(\mathfrak{f}_\alpha)) \subseteq \mathcal{N}(\mathfrak{j})_{\leq t}$  with  $t = \max\{m, |\alpha|\}$ .

For a fixed  $[\mathfrak{j}, m]$ -marked set  $\mathfrak{G}$ , a  $[\mathfrak{j}, m]$ -completion (or shortly, a completion) is a subset  $\bar{\mathfrak{G}} \subset (\mathfrak{G})$  of marked polynomials  $\mathfrak{f}_\beta$  such that the head terms  $\text{Ht}(\mathfrak{f}_\beta) = x^\beta$  are pairwise different, they form the set of all monomials in  $\mathfrak{j}_{\leq m} \setminus B_{\mathfrak{j}}$  and  $\text{supp}(\text{Ht}(\mathfrak{f}_\beta) - \mathfrak{f}_\beta) \subseteq \mathcal{N}(\mathfrak{j})_{\leq m}$ . For every  $\mathfrak{f}_\beta$  belonging to a  $[\mathfrak{j}, m]$ -completion  $\bar{\mathfrak{G}}$  of  $\mathfrak{G}$ , we use again the expression *tail* for the polynomial  $T(\mathfrak{f}_\beta) := \text{Ht}(\mathfrak{f}_\beta) - \mathfrak{f}_\beta$ .

**Definition 4.7.** Given a strongly stable ideal  $\mathfrak{j}$  in  $R$  and a  $[\mathfrak{j}, m]$ -marked set  $\mathfrak{G}$ , consider  $\mathfrak{g} \in R$ . A  $[\mathfrak{j}, m]$ -reduced form modulo  $(\mathfrak{G})$  of  $\mathfrak{g}$  is a polynomial  $\bar{\mathfrak{g}}$  such that  $\text{supp}(\bar{\mathfrak{g}}) \subset \mathcal{N}(\mathfrak{j})_{\leq t}$ , with  $t = \max\{m, \deg(\mathfrak{g})\}$ , and  $\mathfrak{g} - \bar{\mathfrak{g}} \in (\mathfrak{G})$ . If such a  $[\mathfrak{j}, m]$ -reduced form modulo  $(\mathfrak{G})$  exists and is unique for every  $\mathfrak{g} \in R$ , we call it the  $[\mathfrak{j}, m]$ -normal form and denote it by  $\text{Nf}(\mathfrak{g})$ .

We now show that a  $[\mathfrak{j}, m]$ -marked set, with a completion, generates an ideal  $\mathfrak{i}$  such that the quotient  $R/\mathfrak{i}$  in degree  $\leq t$  is generated as a  $K$ -vector space by  $\mathfrak{j}_{\leq t}$ , for every  $t \geq m$ .

**Proposition 4.8.** *Let  $\mathfrak{j}$  be a strongly stable ideal in  $R$  and  $\mathfrak{G}$  be a  $[\mathfrak{j}, m]$ -marked set, for some positive integer  $m$ . Then there exists a  $[\mathfrak{j}, m]$ -completion of  $\mathfrak{G}$  if and only if every  $\mathfrak{g} \in R$  has a  $[\mathfrak{j}, m]$ -reduced form modulo  $(\mathfrak{G})$ .*

*Proof.* Suppose that every  $\mathfrak{g} \in R$  has a  $[j, m]$ -reduced form modulo  $(\mathfrak{G})$ . We can construct a  $[j, m]$ -completion of  $\mathfrak{G}$  taking for every  $x^\beta \in \mathfrak{j}_{\leq m} \setminus B_j$ , the marked polynomial  $x^\beta - \mathfrak{g}_\beta$ ,  $\text{Ht}(x^\beta - \mathfrak{g}_\beta) := x^\beta$ , where  $\mathfrak{g}_\beta$  is a  $[j, m]$ -reduced form modulo  $(\mathfrak{G})$  of  $x^\beta$ , because  $\deg(\mathfrak{g}) = |\beta| \leq m$ .

Vice versa, we now assume that the  $[j, m]$ -marked set  $\mathfrak{G}$  has a completion  $\overline{\mathfrak{G}}$ . It is sufficient to prove that every monomial in  $R$  has a  $[j, m]$ -reduced form modulo  $(\mathfrak{G})$ .

Let  $E$  be the set of monomials in  $R$  which do not have a  $[j, m]$ -reduced form modulo  $(\mathfrak{G})$ . Observe that  $E \cap \mathcal{N}(\mathfrak{j}) = \emptyset$  because if a monomial does not belong to  $\mathfrak{j}$ , then it is its  $[j, m]$ -reduced form modulo  $(\mathfrak{G})$ . Furthermore  $E \cap \mathfrak{j}_{\leq m} = \emptyset$ , because every  $x^\beta \in \mathfrak{j}_{\leq m}$  has a  $[j, m]$ -reduced form modulo  $(\mathfrak{G})$ : indeed, there is  $\mathfrak{f}_\beta \in \mathfrak{G} \cup \overline{\mathfrak{G}}$  such that  $\text{supp}(x^\beta - \mathfrak{f}_\beta) \subseteq \mathcal{N}(\mathfrak{j})_{\leq m}$ .

Suppose  $E$  is not empty and consider  $x^\beta \in E$  with minimum degree  $\deg(x^\beta) = t$  and with minimal  $\min(x^\beta) =: x_i$  among the monomials of  $E$  of degree  $t$ . Note that  $x^\beta$  belongs to  $\mathfrak{j}$  and its degree  $t$  is  $\geq m + 1$ . The monomial  $x^\beta$  can be written as  $x^{\beta'} x_i$ , with  $x^{\beta'} \in \mathfrak{j}_{\leq t-1}$  and  $x_i = \min(x^\beta)$  (by [5, Lemma 1.2]). By the minimality of  $t$  in  $E$ ,  $x^{\beta'}$  has a  $[j, m]$ -reduced form modulo  $(\mathfrak{G})$ , that we denote by  $\mathfrak{g}$ :  $x^{\beta'} - \mathfrak{g} \in (\mathfrak{G})$  and  $\text{supp}(\mathfrak{g}) \subseteq \mathcal{N}(\mathfrak{j})_{\leq t-1}$ . For every monomial  $x_i x^\gamma \in \text{supp}(x_i \mathfrak{g}) \cap \mathfrak{j}$ , since  $x^\gamma \in \text{supp}(\mathfrak{g}) \subseteq \mathcal{N}(\mathfrak{j})_{\leq t-1}$ , we have  $x_i x^\gamma = x^{\alpha'} x_\ell$ , with  $x^{\alpha'} \in \mathfrak{j}$  and  $x_\ell < x_i$  (by Lemma 2.3). By the minimality of  $x_i$ , every monomial of  $\text{supp}(x_i \mathfrak{g}) \cap \mathfrak{j}$  has a  $[j, m]$ -reduced form modulo  $(\mathfrak{G})$ . This is a contradiction, so  $E$  is empty.  $\square$

**Definition 4.9.** Let  $\mathfrak{j}$  be a strongly stable ideal in  $R$  and  $\mathfrak{G}$  be a  $[j, m]$ -marked set, for some positive integer  $m$ .  $\mathfrak{G}$  is a  $[j, m]$ -marked basis if  $R_{\leq t} = \langle \mathcal{N}(\mathfrak{j})_{\leq t} \rangle \oplus (\mathfrak{G})_{\leq t}$  for all  $t \geq m$  as  $K$ -vector spaces.

**Theorem 4.10.** Let  $\mathfrak{j}$  be a strongly stable ideal in  $R$   $m$  be a positive integer and  $\mathfrak{G}$  be a  $[j, m]$ -marked set.  $\mathfrak{G}$  is a  $[j, m]$ -marked basis if and only if there is a  $[j, m]$ -completion  $\overline{\mathfrak{G}}$  and the  $[j, m]$ -reduced forms modulo  $(\mathfrak{G})$  are unique.

*Proof.* If  $\mathfrak{G}$  is a  $[j, m]$ -marked basis, by the definition we have  $R_{\leq t} = \langle \mathcal{N}(\mathfrak{j})_{\leq t} \rangle \oplus (\mathfrak{G})_{\leq t}$  for every  $t \geq m$ . For every polynomial  $\mathfrak{g} \in R$ , there is a unique couple of polynomials  $\mathfrak{g}_1 \in (\mathfrak{G})_{\leq t}$ ,  $\mathfrak{g}_2 \in \langle \mathcal{N}(\mathfrak{j})_{\leq t} \rangle$  such that  $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ , with  $t = \max\{m, \deg(\mathfrak{g})\}$ . Then,  $\mathfrak{g}_2$  is a  $[j, m]$ -reduced form modulo  $(\mathfrak{G})$  of  $\mathfrak{g}$ . By Proposition 4.8, the existence of a  $[j, m]$ -reduced forms modulo  $(\mathfrak{G})$  is equivalent to the existence of a completion, which can be explicitly constructed as in the proof of Proposition 4.8. For what concerns uniqueness, if  $\mathfrak{g} \in R$  has two  $[j, m]$ -reduced forms modulo  $(\mathfrak{G})$ , then their difference belongs to  $\langle \mathcal{N}(\mathfrak{j})_{\leq t} \rangle \cap (\mathfrak{G})_{\leq t} = \{0\}$ , and so they are the same.

Vice versa, we can apply Proposition 4.8 in order to get  $R_{\leq t} = \langle \mathcal{N}(\mathfrak{j})_{\leq t} \rangle + (\mathfrak{G})_{\leq t}$  as  $K$ -vector spaces for every  $t \geq m$ ; by the uniqueness of  $[j, m]$ -reduced form modulo  $(\mathfrak{G})$ , we get the direct sum.  $\square$

**Corollary 4.11.** Let  $\mathfrak{j}$  be a strongly stable ideal in  $R$  and  $\mathfrak{G}$  be a  $[j, m]$ -marked set, for some positive integer  $m$ , with  $\overline{\mathfrak{G}}$  a completion. Then  $\mathcal{N}(\mathfrak{j})_{\leq t}$  generates  $R_{\leq t}/(\mathfrak{G})_{\leq t}$  as  $K$ -vector space and  $\dim_K \mathfrak{j}_{\leq t} \leq \dim_K (\mathfrak{G})_{\leq t}$ , for every  $t \geq m$ .

If  $\mathfrak{G}$  is a  $[j, m]$ -marked basis, then  $\mathcal{N}(\mathfrak{j})_{\leq t}$  is a basis of  $R_{\leq t}/(\mathfrak{G})_{\leq t}$  as a  $K$ -vector space and  $\dim_K \mathfrak{j}_{\leq t} = \dim_K (\mathfrak{G})_{\leq t}$  for all  $t \geq m$ .

*Proof.* If  $\mathfrak{G}$  is a  $[j, m]$ -marked set with  $\overline{\mathfrak{G}}$  a completion then by Proposition 4.8 every monomial in  $\mathfrak{j}$  has a  $[j, m]$ -reduced form modulo  $(\mathfrak{G})$ , so every  $\mathfrak{g}$  in  $R_{\leq t}/(\mathfrak{G})_{\leq t}$  is a linear combination of monomials in  $\mathcal{N}(\mathfrak{j})_{\leq t}$ , for every  $t \geq m$ . Then  $\mathcal{N}(\mathfrak{j})_{\leq t}$  generates the  $K$ -vector space  $R_{\leq t}/(\mathfrak{G})_{\leq t}$ . If  $\mathfrak{G}$  is a  $[j, m]$ -marked basis then every  $\mathfrak{g}$  in  $R_{\leq t}/(\mathfrak{G})_{\leq t}$  is a linear combination of monomials in  $\mathcal{N}(\mathfrak{j})_{\leq t}$  and such a linear combination is unique, by Theorem 4.10. Then  $\mathcal{N}(\mathfrak{j})_{\leq t}$  is a basis of  $R_{\leq t}/(\mathfrak{G})_{\leq t}$  for all  $t \geq m$ .  $\square$

We end this subsection with two properties of  $[j, m]$ -marked bases. The first one establishes that, given a  $[j, m]$ -marked basis  $\mathfrak{G}$ , every polynomial  $\mathfrak{g}$  in  $R$  is in  $\mathfrak{G}_*$ -relation with its  $[j, m]$ -normal form  $\text{Nf}(\mathfrak{g})$ , that exists by Theorem 4.10. Hence, the  $\mathfrak{G}_*$ -reduction gives us a division algorithm to compute exactly  $\text{Nf}(\mathfrak{g})$ . The second property is similar to a feature of border bases [28, 29].

**Proposition 4.12.** *Let  $j$  be a strongly stable ideal in  $R$ ,  $m$  be a positive integer and  $\mathfrak{G}$  be a  $[j, m]$ -marked basis. Then:*

- (i) *for every  $\mathfrak{g} \in R$ , the polynomial  $\text{Nf}(\mathfrak{g})$  is the reduced polynomial we obtain by applying  $\xrightarrow{\mathfrak{G}_*}$  to  $\mathfrak{g}$ ;*
- (ii) *for every  $x^\beta \in R$ , for every  $x_i$ ,  $\text{Nf}(x_i x^\beta) = \text{Nf}(x_i \text{Nf}(x^\beta))$ .*

*Proof.*

- (i) We consider  $\mathfrak{g} \in R$  and we compute  $\mathfrak{g}_1$ , the reduced polynomial we obtain from  $\mathfrak{g}$  by  $\xrightarrow{\mathfrak{G}_*}$ :  $\text{supp}(\mathfrak{g}_1)$  is a subset of  $\mathcal{N}(j)$ . We define  $\bar{t} := \max\{\deg(\mathfrak{g}), \deg(\mathfrak{g}_1), m\}$  and consider  $\text{Nf}(\mathfrak{g})$ :  $\mathfrak{g}_1 - \text{Nf}(\mathfrak{g})$  belongs to  $(\mathfrak{G})_{\leq \bar{t}}$  and its support is contained in  $\mathcal{N}(j)_{\leq \bar{t}}$ . Since  $\mathfrak{G}$  is a  $[j, m]$ -marked basis,  $\mathfrak{g}_1 = \text{Nf}(\mathfrak{g})$ .
- (ii) By the previous item, for every  $x^\beta \in R$ ,  $x^\beta \xrightarrow{\mathfrak{G}_*} \text{Nf}(x^\beta)$ ,  $x_i x^\beta \xrightarrow{\mathfrak{G}_*} \text{Nf}(x_i x^\beta)$  and  $x_i \text{Nf}(x^\beta) \xrightarrow{\mathfrak{G}_*} \text{Nf}(x_i \text{Nf}(x^\beta))$ .

Observe that  $\mathfrak{g}_1 := x^\beta - \text{Nf}(x^\beta)$ ,  $\mathfrak{g}_2 := x_i x^\beta - \text{Nf}(x_i x^\beta)$  and  $\mathfrak{g}_3 := x_i \text{Nf}(x^\beta) - \text{Nf}(x_i \text{Nf}(x^\beta))$  all belong to  $(\mathfrak{G})$  and the supports of  $\text{Nf}(x^\beta)$  and  $\text{Nf}(x_i x^\beta)$  are contained in  $\mathcal{N}(j)$ . Then

$$x_i \mathfrak{g}_1 - \mathfrak{g}_2 + \mathfrak{g}_3 = \text{Nf}(x_i x^\beta) - \text{Nf}(x_i \text{Nf}(x^\beta)) \in (\mathfrak{G})_{\leq t} \cap \langle \mathcal{N}(j)_{\leq t} \rangle \text{ for some integer } t.$$

Since  $\mathfrak{G}$  is a  $[j, m]$ -marked basis, we obtain  $\text{Nf}(x_i x^\beta) = \text{Nf}(x_i \text{Nf}(x^\beta))$ .

□

**4.2. Marked sets in  $S$  and in  $R$ .** Given a strongly stable ideal  $j \subset R$  and a positive integer  $m$ , in the previous section we have introduced the notion of  $[j, m]$ -marked set  $\mathfrak{G}$  endowed with a completion and, in particular, that of  $[j, m]$ -marked basis. We have also investigated main properties of them and how these notions are related with the reduction procedure  $\xrightarrow{\mathfrak{G}_*}$ .

Here we will show that the strong similarity among the above notions and those of  $J_{\geq m}$ -marked sets, bases and superminimal reduction, for  $J$  saturated strongly stable ideal in  $S$  (see [12, Theorem 2.2, Corollary 2.4], [5, Theorem 3.14]), conveys a precise correspondence we are going to investigate.

We define two special integers that will be useful in the next statements and proofs:

- for every  $x^\alpha \in S$ , we define  $t_\alpha := \max\{0, m - |\alpha|\}$ ;
- for every  $\mathfrak{g} \in R$ , we define  $m_g := \max\{0, m - \deg \mathfrak{g}\}$ . If we consider a marked polynomial  $f_\alpha \in R$ , we simply write  $m_\alpha$  instead of  $m_{f_\alpha}$ .

Recall that, given an ideal  $\mathfrak{a} \subset R$ , its homogenization is the ideal  $\mathfrak{a}^h \subset S$  generated by the homogenizations  $f^h$  of all the polynomials  $f$  of  $\mathfrak{a}$ . In general, if we homogenize a set of generators of  $\mathfrak{a} \subset R$ , we do not get a set of generators of  $\mathfrak{a}^h$ . However, the situation is far simpler for monomial ideals, and in particular for a strongly stable ideal  $j$  of  $R$  with monomial basis  $B_j$ : its homogenization is exactly the ideal  $j^h = (B_j)S$ . Vice versa, given a homogeneous ideal  $A$  in  $S$  generated by a finite set of homogeneous polynomials  $\{F_1, \dots, F_s\}$ , the ideal  $A^a$  in  $R$  is generated by the set  $\{F_1^a, \dots, F_s^a\}$ . For a saturated strongly stable ideal  $J$  we get the strongly stable ideal  $J^a = (B_J)R$ .

We now state a quite simple fact concerning the relation between the sous-escaliers of strongly stable ideals when we homogenize or dehomogenize them. This will be useful when considering homogenizations and dehomogenizations of marked sets and bases.

**Proposition 4.13.**

- (i) Let  $\mathfrak{j}$  be a strongly stable ideal in  $R$ . The monomial  $x^\gamma$  belongs to  $\mathcal{N}(\mathfrak{j})$  if and only if for every  $r \geq 0$ ,  $x_0^r x^\gamma$  belongs to  $\mathcal{N}(\mathfrak{j}^h)$ .
- (ii) Let  $J$  be a saturated strongly stable ideal in  $S$ . The monomial  $x^\gamma$  belongs to  $\mathcal{N}(J)$  if and only if  $(x^\gamma)^a$  belongs to  $\mathcal{N}(J^a)$ .

*Proof.* The statements are consequences of the facts that  $B_{\mathfrak{j}} = B_{\mathfrak{j}^h}$  and, if  $J$  is saturated, that  $B_J = B_{J^a}$ .  $\square$

**Corollary 4.14.**

- (1) Let  $\mathfrak{j}$  be a strongly stable ideal in  $R$  and  $\mathfrak{f}$  a polynomial in  $R$  with  $\deg(\mathfrak{f}_\alpha) = t$ . Then

$$\text{supp}(\mathfrak{f}) \subseteq \mathcal{N}(\mathfrak{j})_{\leq t} \Leftrightarrow \text{for every } r \geq 0, \text{supp}(x_0^r \mathfrak{f}^h) \subseteq \mathcal{N}(\mathfrak{j}^h)_{t+r}.$$

- (2) Let  $J$  be a saturated strongly stable ideal in  $S$  and  $f$  a homogeneous polynomial in  $S$ . Then

$$\text{supp}(f) \subseteq \mathcal{N}(J)_t \Leftrightarrow \text{supp}(f^a) \subseteq \mathcal{N}(J^a)_{\leq t}.$$

**Remark 4.15.** In the same setting of Corollary 4.14, take a marked polynomial  $\mathfrak{f} = \text{Ht}(\mathfrak{f}) - T(\mathfrak{f}) \in R$ , with  $\text{Ht}(\mathfrak{f}) \in \mathfrak{j}$  and  $T(\mathfrak{f}) \in \mathcal{N}(\mathfrak{j})$ . Then, its homogenization  $\mathfrak{f}^h \in S$  is a marked polynomial over  $\mathfrak{j}^h$  with  $\text{Ht}(\mathfrak{f}^h) = x_0^{\deg(\mathfrak{f}) - \deg(\text{Ht}(\mathfrak{f}))} \text{Ht}(\mathfrak{f})$ ,  $T(\mathfrak{f}^h) = x_0^{\deg(\mathfrak{f}) - \deg(T(\mathfrak{f}))} T(\mathfrak{f})^h$  and  $\text{supp}(T(\mathfrak{f}^h)) \subseteq \mathcal{N}(\mathfrak{j}^h)$ . Further, given a saturated strongly stable ideal  $J$  of  $S$ , a positive integer  $m$  and a marked polynomial  $f = \text{Ht}(f) - T(f) \in S$  with  $\text{Ht}(f) \in J_{\geq m}$  and  $\text{supp}(T(f)) \subseteq \mathcal{N}(J)$ , let  $J^a$  be the strongly stable ideal generated in  $R$  by the monomial basis of  $J$ . Then the polynomial  $f^a \in R$  is marked over  $J^a$  with  $\text{Ht}(f^a) = \text{Ht}(f)^a$  and  $T(f^a) = T(f)^a$ .

**Lemma 4.16.** Let  $\mathfrak{j}$  be a strongly stable ideal in  $R$ ,  $m$  be a positive integer,  $\mathfrak{G} \subset R$  be a  $[\mathfrak{j}, m]$ -marked set and let  $\overline{\mathfrak{G}}$  be a  $[\mathfrak{j}, m]$ -completion of  $\mathfrak{G}$ . Then, the following set of homogeneous marked polynomials in  $S$

$$(4.1) \quad G = \{x_0^{m_\alpha} \mathfrak{f}_\alpha^h | \mathfrak{f}_\alpha \in \mathfrak{G} \cup \overline{\mathfrak{G}}\}, \text{ where } \text{Ht}(x_0^{m_\alpha} \mathfrak{f}_\alpha^h) = x_0^{m_\alpha + \deg(\mathfrak{f}_\alpha) - |\alpha|} x^\alpha$$

is a  $\mathfrak{j}_{\geq m}^h$ -marked set, with  $\mathfrak{j}_{\geq m}^h$ -marked superminimal set  $sG = \{x_0^{m_\alpha} \mathfrak{f}_\alpha^h | \mathfrak{f}_\alpha \in \mathfrak{G}\}$ .

*Proof.* Observe that the head terms of the marked polynomials in  $G$  as defined in (4.1) are pairwise different and they constitute the monomial basis of  $\mathfrak{j}_{\geq m}^h$  in  $S$ ; further, the tails of these polynomials are contained in  $\mathcal{N}(\mathfrak{j}_{\geq m}^h)$  by Corollary 4.14, hence  $G$  is a  $\mathfrak{j}_{\geq m}^h$ -marked set in  $S$ . Furthermore, the head terms of the polynomials obtained from the marked polynomials in  $\mathfrak{G}$  are exactly of kind  $x_0^{t_\alpha} x^\alpha$ ,  $x^\alpha \in B_{\mathfrak{j}}$ . These monomials are exactly the superminimal generators of  $\mathfrak{j}_{\geq m}^h$ , hence  $sG = \{x_0^{m_\alpha} \mathfrak{f}_\alpha^h | \mathfrak{f}_\alpha \in \mathfrak{G}\}$ .  $\square$

**Lemma 4.17.** Let  $J$  be a saturated strongly stable ideal in  $S$ ,  $m$  be a positive integer,  $G$  be a  $J_{\geq m}$ -marked set in  $S$  and let  $sG$  be its  $J_{\geq m}$ -marked superminimal set. Then the following set of marked polynomials in  $R$

$$(4.2) \quad \mathfrak{G} = \{F_\alpha^a | F_\alpha \in sG\}, \text{ where } \text{Ht}(F_\alpha^a) := \text{Ht}(F_\alpha)^a$$

is a  $[J^a, m]$ -marked set, having a completion defined as

$$(4.3) \quad \overline{\mathfrak{G}} = \{F_\beta^a | F_\beta \in G \setminus sG\}, \text{ where } \text{Ht}(F_\beta^a) := \text{Ht}(F_\beta)^a.$$

*Proof.* The head terms of the marked polynomials in  $\mathfrak{G}$  as defined in (4.2) are pairwise different and they constitute the monomial basis of  $J^a$ . Furthermore the head terms of the polynomials in  $\overline{\mathfrak{G}}$  constitute the set of monomials in  $J^a_{\leq m} \setminus B_J$ . Finally, the tails of the marked polynomials  $F_\alpha^a$  are supported on  $\mathcal{N}(J^a)$  by Corollary 4.14 and their degrees are bounded by  $\max\{m, \deg(\text{Ht}(F_\alpha^a))\}$ . Then  $\mathfrak{G}$  is a  $[J^a, m]$ -marked set having  $\overline{\mathfrak{G}}$  as completion.  $\square$

**Theorem 4.18.**

- (1) Let  $\mathfrak{j}$  be a strongly stable ideal in  $R$ ,  $m$  be a positive integer,  $\mathfrak{G} \subset R$  be a  $[\mathfrak{j}, m]$ -marked basis. Then the  $\mathfrak{j}_{\geq m}^h$ -marked set  $G \subset S$ , as constructed in Lemma 4.16, is a  $\mathfrak{j}_{\geq m}^h$ -marked basis.
- (2) Let  $J$  be a saturated strongly stable ideal in  $S$ ,  $m$  be a positive integer,  $G \subset S$  be a  $J_{\geq m}$ -marked basis. Then the  $[J^a, m]$ -marked set  $\mathfrak{G} \subset R$ , as constructed in Lemma 4.17, is a  $[J^a, m]$ -marked basis.

*Proof.*

- (1)  $\mathfrak{G} \subset R$  is a  $[\mathfrak{j}, m]$ -marked basis, so there is a completion  $\overline{\mathfrak{G}}$ , by Theorem 4.10. Using  $\mathfrak{G}$  and  $\overline{\mathfrak{G}}$ , we construct  $G$  as in Lemma 4.16:  $G$  is a  $\mathfrak{j}_{\geq m}^h$ -marked set, hence, by [12, Corollary 2.3],  $\mathcal{N}(\mathfrak{j}_{\geq m}^h)$  generates  $S/(G)$  as  $K$ -vector space:  $S_t = \mathcal{N}(\mathfrak{j}_{\geq m}^h)_t + (G)_t$  for every  $t$ . We now show that  $\langle \mathcal{N}(\mathfrak{j}_{\geq m}^h) \rangle \cap (G) = \{0\}$ .

Consider a homogeneous polynomial  $g \in \langle \mathcal{N}(\mathfrak{j}_{\geq m}^h) \rangle \cap (G)_t$ . Thanks to the construction of  $G$ ,  $g^a$  is a polynomial belonging to  $(\mathfrak{G})_{\leq t}$  whose support is contained in  $\mathcal{N}(\mathfrak{j})_{\leq t}$ ; thus,  $g^a \in \langle \mathcal{N}(\mathfrak{j})_{\leq t} \rangle \cap (\mathfrak{G})_{\leq t}$  and  $g^a$  is the null polynomial, by Definition 4.9. This implies that  $g$  is the null polynomial too. Then  $G$  is a  $\mathfrak{j}_{\geq m}^h$ -marked basis, by Definition 2.6.

- (2)  $G \subset S$  is a  $J_{\geq m}$ -marked basis and we construct  $\mathfrak{G}$  and  $\overline{\mathfrak{G}}$  as in Lemma 4.17:  $\mathfrak{G}$  is a  $[J^a, m]$ -marked set and  $\overline{\mathfrak{G}}$  a completion. By Corollary 4.11,  $\mathcal{N}(J^a)_{\leq t}$  generates  $R_{\leq t}/(\mathfrak{G})_{\leq t}$  as  $K$ -vector space for every  $t \geq m$ , hence  $R_{\leq t} = \mathcal{N}(J^a)_{\leq t} + (\mathfrak{G})_{\leq t}$ . We now prove that  $\mathcal{N}(J^a)_{\leq t} \cap (\mathfrak{G})_{\leq t} = \{0\}$  for every  $t \geq m$ .

Consider  $\mathfrak{g} \in \mathcal{N}(J^a)_{\leq t} \cap (\mathfrak{G})_{\leq t}$ ,  $t = \max\{m, \deg(g)\}$ . Thanks to the construction of  $\mathfrak{G}$  and  $\overline{\mathfrak{G}}$ , the polynomial  $x_0^{m_{\mathfrak{g}}} \mathfrak{g}^h$  belongs to  $(G)_t$  and its support is contained in  $\mathcal{N}(J_{\geq m})_t$ . By Definition 2.6,  $x_0^{m_{\mathfrak{g}}} \mathfrak{g}^h$  is the null polynomial in  $S$ , hence  $\mathfrak{g}$  is the null polynomial in  $R$ . Then  $\mathfrak{G}$  is a  $[J^a, m]$ -marked basis, by Definition 4.9. □

**Proposition 4.19.** Let  $\mathfrak{j}$  be a strongly stable ideal in  $R$ ,  $m$  be a positive integer,  $\mathfrak{G} \subset R$  be a  $[\mathfrak{j}, m]$ -marked set with  $\overline{\mathfrak{G}} = \{f_{\beta}\}$  a  $[\mathfrak{j}, m]$ -completion. Let  $G \subset S$  be the  $\mathfrak{j}_{\geq m}^h$ -marked set constructed from  $\mathfrak{G} \cup \overline{\mathfrak{G}}$  as in Lemma 4.16.

Consider  $\mathfrak{g} \in R$ ,  $\mathfrak{g} \xrightarrow{\mathfrak{G}^*} \mathfrak{g}_1$ . Then there is  $t_0$  such that for every  $t \geq t_0$ ,  $x_0^t \mathfrak{g}^h \xrightarrow{sG^*} x_0^{t_1} \mathfrak{g}_1^h$  with  $t + \deg(\mathfrak{g}) = t_1 + \deg(\mathfrak{g}_1)$ .

*Proof.* We prove that the thesis holds for each step of reduction. Let  $\mathfrak{g}$  be a polynomial in  $R$ . Consider  $x^{\epsilon} \in \text{supp}(\mathfrak{g}) \cap \mathfrak{j}$  and  $x^{\epsilon} = x^{\alpha} *_{\mathfrak{j}} x^{\delta}$ . In order to reduce  $x^{\epsilon}$ , we perform the step  $\mathfrak{g} \xrightarrow{\mathfrak{G}^*} \mathfrak{g} - c_{\epsilon} x^{\delta} f_{\alpha}$ , where  $c_{\epsilon}$  is the coefficient of  $x^{\epsilon}$  in  $\mathfrak{g}$ . Observe that:

$$(\mathfrak{g} - c_{\epsilon} x^{\delta} f_{\alpha})^h = \begin{cases} \mathfrak{g}^h - c_{\epsilon} x_0^{\deg(\mathfrak{g}) - \deg(x^{\delta} f_{\alpha})} x^{\delta} f_{\alpha}^h & \text{if } \deg(\mathfrak{g}) \geq \deg(x^{\delta} f_{\alpha}) \\ x_0^{\deg(x^{\delta} f_{\alpha}) - \deg(\mathfrak{g})} \mathfrak{g}^h - c_{\epsilon} x^{\delta} f_{\alpha}^h & \text{otherwise} \end{cases}$$

We now show that there is  $t_0$  such that for every  $t \geq t_0$ ,  $x_0^t \mathfrak{g}^h \xrightarrow{sG^*} x_0^{t_1} (\mathfrak{g} - c_{\epsilon} x^{\delta} f_{\alpha})^h$ , where these two homogeneous polynomials have the same degree.

Since  $x^{\alpha} \in B_{\mathfrak{j}^h}$ , there is  $x_0^{m_{\alpha}} f_{\alpha}^h \in sG$  such that  $\text{Ht}(x_0^{m_{\alpha}} f_{\alpha}^h) = x_0^{t_{\alpha}} x^{\alpha} \in sB_{\mathfrak{j}^h}$ . We will apply a step of superminimal reduction using  $x_0^{m_{\alpha}} f_{\alpha}^h \in sG$ . In  $\mathfrak{g}^h$ , the monomial  $x^{\epsilon}$  becomes the monomial  $x_0^{\deg(\mathfrak{g}) - |\epsilon|} x^{\epsilon} = x^{\alpha} *_{\mathfrak{j}^h} (x^{\delta} x_0^{\deg(\mathfrak{g}) - |\epsilon|}) \in \text{supp}(\mathfrak{g}^h) \cap \mathfrak{j}_{\geq m}^h$ . In  $x^{\delta} f_{\alpha}^h$ , the monomial  $x^{\epsilon}$  becomes  $x_0^{\deg(f_{\alpha}) - |\alpha|} x^{\delta} x^{\alpha}$ .

If  $\deg(\mathfrak{g}) \geq \deg(x^{\delta} f_{\alpha})$ , we define  $r := \max\{0, m_{\alpha} - \deg(g) + \deg(x^{\delta} f_{\alpha}^h)\}$ . The monomial  $x_0^r$  is the smallest power of  $x_0$  allowing to perform a step of superminimal reduction by  $x_0^{m_{\alpha}} f_{\alpha}^h$ .



Reducing by  $\xrightarrow{sG_*}$ , we obtain:

$$x_0^r \mathfrak{g}^h \xrightarrow{sG_*} x_0^r \mathfrak{g}^h - c_\epsilon x_0^{r+\deg(\mathfrak{g})-\deg(x^\delta \mathfrak{f}_\alpha)} x^\delta \mathfrak{f}_\alpha^h = x_0^r (\mathfrak{g} - c_\epsilon x^\delta \mathfrak{f}_\alpha)^h.$$

We observe that in this case  $t_0 = r = \max\{0, m_\alpha - \deg(\mathfrak{g}) + \deg(x^\delta \mathfrak{f}_\alpha)\}$  and for every  $t \geq t_0$ ,  $t_1 = t$ .

If  $\deg(\mathfrak{g}) < \deg(x^\delta \mathfrak{f}_\alpha)$ , we define  $r := m_\alpha + \deg(x^\delta \mathfrak{f}_\alpha) - \deg(\mathfrak{g})$ . Again, in this case the monomial  $x_0^r$  is the smallest power of  $x_0$  allowing to perform a step of superminimal reduction by  $x_0^{m_\alpha} \mathfrak{f}_\alpha^h$ . Reducing by  $\xrightarrow{sG_*}$ , we obtain:

$$x_0^r \mathfrak{g}^h \xrightarrow{sG_*} x_0^r \mathfrak{g}^h - c_\epsilon x_0^{m_\alpha} x^\delta \mathfrak{f}_\alpha^h = x_0^{m_\alpha} (\mathfrak{g} - c_\epsilon x^\delta \mathfrak{f}_\alpha)^h.$$

We observe that in this case  $t_0 = r = m_\alpha + \deg(x^\delta \mathfrak{f}_\alpha) - \deg(\mathfrak{g})$  and for every  $t \geq t_0$ ,  $t_1 = t - \deg(x^\delta \mathfrak{f}_\alpha) + \deg(\mathfrak{g})$ .  $\square$

**Proposition 4.20.** *Let  $J$  be a saturated strongly stable ideal in  $S$ ,  $m$  be a positive integer and  $G$  be a  $J_{\geq m}$ -marked set. Let  $\mathfrak{G} \subset R$  be the  $[J^a, m]$ -marked set constructed from  $G$  as in Lemma 4.17.*

*If  $g$  is a homogeneous polynomial belonging to  $S$  and there is  $t$  such that  $x_0^t g \xrightarrow{sG_*} g_1$  then  $g^a \xrightarrow{\mathfrak{G}_*} g_1^a$ .*

*Proof.* We prove that the thesis holds for each step of reduction.

Consider  $x^\epsilon \in \text{supp}(g) \cap J$ :  $x^\epsilon = x^\alpha *_J x^\delta$ . If  $|\alpha| \geq m$ , then the reduction step is  $g \xrightarrow{sG_*} g - c_\epsilon x^\delta F_\alpha$ . Correspondingly, if we consider  $(x^\epsilon)^a \in \text{supp}(g^a) \cap J^a$ , then  $(x^\epsilon)^a = x^\alpha *_J (x^\delta)^a$ , and so the reduction in the non-homogeneous case is exactly  $g^a \xrightarrow{\mathfrak{G}_*} g^a - c_\epsilon (x^\delta)^a F_\alpha^a$ .

If  $|\alpha| < m$ , then the reduction step is  $x_0^t g \xrightarrow{sG_*} x_0^t g - c_\epsilon x^{\delta'} F_{\alpha'}$  for a suitable power  $t$  such that  $x^{\alpha'} = x^\alpha x_0^{m-|\alpha|}$  and  $x^{\delta'} = x^\delta x_0^{t-m+|\alpha|}$ . Again, if we consider  $(x^\epsilon)^a$ , it decomposes as  $x^\alpha *_J (x^\delta)^a = (x^{\alpha'})^a *_J (x^{\delta'})^a$ , hence the reduction in the non-homogeneous case is exactly

$$g^a \xrightarrow{\mathfrak{G}_*} g^a - c_\epsilon (x^{\delta'})^a (F_{\alpha'})^a = (x_0^t g - c_\epsilon x^{\delta'} F_{\alpha'})^a.$$

$\square$

Recall that, for suitable term orders, Gröbner bases behave well with respect to the homogenization, in the sense that the homogenizations of the polynomials of such a Gröbner basis  $Gb$  generate the homogenization of the ideal  $(Gb)$  (for example, see [13, Chapter 8]). Theorem 4.18 shows that  $[j, m]$ -marked basis in  $R$  have an analogous nice behaviour.

**Proposition 4.21.** *Let  $\mathfrak{j}$  be a strongly stable ideal in  $R$ ,  $m$  be a positive integer,  $\mathfrak{G}$  be a  $[j, m]$ -marked basis. Then  $(\mathfrak{G})_{\geq m}^h = (G)_{\geq m} = (\mathfrak{G}^h \cup \overline{\mathfrak{G}}^h)_{\geq m}$  and  $(\mathfrak{G})^h = (\mathfrak{G}^h \cup \overline{\mathfrak{G}}^h)^{\text{sat}}$ .*

*Proof.* For every ideal in  $R$ , we can obtain its homogenization in  $S$  considering the ideal generated by the homogenization of a set of generators and saturating it by  $x_0$  [25, Corollary 4.3.8]. In our case, we consider the set of generators  $\mathfrak{G} \cup \overline{\mathfrak{G}}$  and obtain  $(\mathfrak{G})_{\geq m}^h = ((\mathfrak{G}^h \cup \overline{\mathfrak{G}}^h) : x_0^\infty)_{\geq m}$ . On the other hand, consider the  $\mathfrak{j}_{\geq m}$ -marked basis  $G$  constructed from  $\mathfrak{G} \cup \overline{\mathfrak{G}}$  in Lemma 4.16. We have that  $(G) \subset S$  belongs to  $\mathcal{Mf}(\mathfrak{j}_{\geq m})$ , hence  $(G)$  is  $m$ -saturated, by Corollary 3.7, and  $((G) : x_0^\infty)_{\geq m} = (G)_{\geq m}$ , by Theorem 3.5.

The inclusion  $((G) : x_0^\infty)_{\geq m} \subseteq ((\mathfrak{G}^h \cup \overline{\mathfrak{G}}^h) : x_0^\infty)_{\geq m}$  is obvious, by the construction of  $G$ . For the other inclusion, consider  $g \in ((\mathfrak{G}^h \cup \overline{\mathfrak{G}}^h) : x_0^\infty)_{\geq m}$ : there is  $t$  such that  $x_0^t g = \sum a_j x^{\delta_j} f_{\alpha_j}^h$ , with  $f_{\alpha_j} \in \mathfrak{G} \cup \overline{\mathfrak{G}}$ . Let  $\overline{m}$  be the maximum in the set of integers  $m_{\alpha_j}$ . Then  $x_0^{t+\overline{m}} g$  belongs to  $(G)$ , hence the other inclusion holds.



We have proved till here that  $(\mathfrak{G})_{\geq m}^h = (G)_{\geq m}$ . Observe now that  $(G)_{\geq m} \subseteq (\mathfrak{G}^h \cup \overline{\mathfrak{G}}^h)_{\geq m}$  by construction and  $(\mathfrak{G}^h \cup \overline{\mathfrak{G}}^h)_{\geq m} \subseteq (\mathfrak{G})_{\geq m}^h$ , by definition of homogenization of an ideal. We get a chain of inclusions which turn out to be equalities, hence  $(\mathfrak{G}^h \cup \overline{\mathfrak{G}}^h)_{\geq m} = (\mathfrak{G})_{\geq m}^h$ . Saturating both homogeneous ideals, we get  $(\mathfrak{G})^h = (\mathfrak{G}^h \cup \overline{\mathfrak{G}}^h)^{\text{sat}}$ .  $\square$

One may think that the homogenization of a  $[j, m]$ -marked basis  $\mathfrak{G}$  with its completion generates  $(\mathfrak{G})^h$ , without saturating. The following example shows that this is not the case.

**Example 4.22.** Consider  $\mathfrak{G} = \{x_2 - x_1^3 + x_1^2 + x_1 + 2, x_1^4 - 2x_1^3 - 2 - x_1\} \subset K[x_1, x_2]$  which is a  $[(x_2, x_1^4), 3]$ -marked basis. We can compute by means of  $\xrightarrow{\mathfrak{G}_*}$  the  $[j, m]$ -reduced forms modulo  $\mathfrak{G}$  of the monomials in  $j_{\leq m} \setminus B_j$  and construct in this way the following polynomials:

$$\begin{aligned} f_1 &:= x_2^3 - 9x_1^3 - 10 - 7x_1 + 21x_1^2, & f_2 &:= x_2^2 x_1 - x_1^3 + 6 + x_1 - 3x_1^2, \\ f_3 &:= x_2^2 + 3x_1^3 - 2 - 3x_1 - 7x_1^2, & f_4 &:= x_2 x_1^2 - x_1^3 - 2 - 3x_1 + x_1^2, & f_5 &:= x_2 x_1 - x_1^3 - 2 + x_1 + x_1^2. \end{aligned}$$

We define the completion  $\overline{\mathfrak{G}}$  as the set  $\{f_1, f_2, f_3, f_4, f_5\}$ . In this case, observe that  $(\mathfrak{G})_{\geq 3}^h$  is equal to  $(\mathfrak{G}^h \cup \overline{\mathfrak{G}}^h)_{\geq 3}$ , but the equality does not hold if we consider the truncation of both ideals from degree 2 on. Indeed, the polynomial  $3 \cdot f_5 + f_3$  has degree 2, its homogenization belongs to  $(\mathfrak{G}^h)$  by definition, however it does not belong to  $(\mathfrak{G}^h \cup \overline{\mathfrak{G}}^h)$ .

**4.3. Effective criterion for  $[j, m]$ -marked bases.** The following theorem will give an algorithmic criterion to test whether a  $[j, m]$ -marked set of polynomials is a  $[j, m]$ -marked basis. The interest of this algorithm is twofold: on the one hand, thanks to the results of Section 4.2, we can avoid the multiplication by powers of  $x_0$ , which represents a bottleneck for the efficiency of the Algorithm presented in [5]; on the other hand, this algorithm will give interesting theoretical results deepening the understanding of the structure of marked families, that we will explore in the Section 4.4.

**Theorem 4.23.** *Let  $j$  be a strongly stable ideal in  $R$  and  $\mathfrak{G}$  be a  $[j, m]$ -marked set, for some positive integer  $m$ .  $\mathfrak{G}$  is a  $[j, m]$ -marked basis if and only if:*

- (i) *for every  $f_\alpha \in \mathfrak{G}$ , for every  $x_i > \min(x^\alpha)$ ,  $x_i f_\alpha \xrightarrow{\mathfrak{G}_*} 0$ ;*
- (ii) *for all  $x^\beta \in j_{\leq m} \setminus B_j$ ,  $x^\beta \xrightarrow{\mathfrak{G}_*} g_\beta$  with  $\text{supp}(g_\beta) \subseteq \mathcal{N}(j)_{\leq m}$ .*

*Proof.* If  $\mathfrak{G}$  is a  $[j, m]$ -marked basis, then there is a completion  $\overline{\mathfrak{G}}$  and every polynomial  $g$  in  $R$  is in  $\mathfrak{G}_*$ -relation with  $\text{Nf}(g)$ , by Proposition 4.12(i).

In the present hypothesis, we have that for every  $x^\beta \in j_{\leq m} \setminus B_j$ , then  $x^\beta \xrightarrow{\mathfrak{G}_*} \text{Nf}(x^\beta)$  with  $\text{supp}(\text{Nf}(x^\beta)) \subset \langle \mathcal{N}(j)_{\leq m} \rangle$  and analogously, for every  $f_\alpha \in \mathfrak{G}$ , for every  $x_i > \min(x^\alpha)$ , the reduction  $\xrightarrow{\mathfrak{G}_*}$  on the polynomial  $x_i f_\alpha$  leads to 0 (by Proposition 4.12(i)).

Vice versa, we now assume that conditions (i) and (ii) hold. By (ii), we define a completion  $\overline{\mathfrak{G}}$  in the following way:

$$\overline{\mathfrak{G}} = \{f_\beta = x^\beta - g_\beta : x^\beta \in j_{\leq m} \setminus B_j, x^\beta \xrightarrow{\mathfrak{G}_*} g_\beta\}, \quad \text{Ht}(f_\beta) = x^\beta.$$

Using  $\mathfrak{G}$  and  $\overline{\mathfrak{G}}$ , by Lemma 4.16 we construct the  $j_{\geq m}^h$ -marked set  $G \subset S$ . By Theorem 4.18, if we prove that  $G$  is a  $j_{\geq m}^h$ -marked basis, we get the thesis. Indeed,  $G$  satisfies the conditions of [5, Proposition 5.5]:

- (1) by Proposition 4.19 and condition (ii), for every  $x^{\beta'} \in B_{j_{\geq m}^h}$ , there exists  $t$  such that

$$x_0^t \cdot x^{\beta'} \xrightarrow{sG_*} g_\beta,$$

where  $x^{\beta'} = x_0^{m-|\beta|} x^\beta$  for some  $x^\beta \in j_{\leq m} \setminus B_j$  and  $g_\beta = x_0^{t_1} (\text{Nf}(x^\beta))^h$ , hence  $\text{supp}(g_\beta) \subset \mathcal{N}(j^h)_{\leq m}$  by Corollary 4.14.

- (2) by Proposition 4.19 and by condition (i), for every polynomial  $F_\alpha \in sG$  and for every  $x_i > \min((x^\alpha)^a)$  there exists  $t$  such that

$$x_0^t x_i F_\alpha \xrightarrow{sG_*} 0.$$

Then  $(G) \subset S$  belongs to  $\mathcal{Mf}(j_{\geq m}^h)$  and  $G$  is a  $j_{\geq m}^h$ -marked basis in  $S$  [5, Proposition 1.11]. Hence  $\mathfrak{G}$  is a  $[j, m]$ -marked basis in  $R$ .  $\square$

The following results highlight another feature of  $[j, m]$ -marked bases analogous to one of Gröbner bases, that involves the notion of syzygy.

**Corollary 4.24.** *Let  $j$  be a strongly stable ideal in  $R$ ,  $m$  be a positive integer and  $\mathfrak{G}$  be a  $[j, m]$ -marked basis. Then every homogeneous syzygy of  $j$  lifts to a syzygy of  $\mathfrak{G}$ .*

*Proof.* For a strongly stable ideal  $j$  in  $R$ , a set of generators for the module of the first syzygies is given by the couples  $(x_i, x^\delta)$  such that  $x_i x^{\alpha_1} - x^\delta x^{\alpha_2} = 0$ , where  $x^{\alpha_1}, x^{\alpha_2} \in B_j$ ,  $x_i > \max(x^{\alpha_1})$  and  $x^\delta x^{\alpha_2} = x^{\alpha_2} *_{j} x^\delta$  [16]. Since  $\mathfrak{G}$  is a  $[j, m]$ -marked basis, for every  $f_\alpha \in \mathfrak{G}$  and every  $x_i > \min(x^\alpha)$ ,  $x_i f_\alpha \xrightarrow{\mathfrak{G}_*} 0$ , by Theorem 4.23(ii). This means that  $x_i f_\alpha = \sum c_i x^{\delta_i} f_{\alpha_i}$ , where  $x^{\delta_i} x^{\alpha_i} = x^{\alpha_i} *_{j} x^{\delta_i}$ . In particular, among the polynomials  $x^{\delta_i} f_{\alpha_i}$  there is  $x^{\delta'} f_{\alpha'}$  such that  $x_i x^\alpha = x^{\alpha'} *_{j} x^{\delta'}$ .  $\square$

One of the key points of Theorem 4.23 is condition (ii) because it allows to define a  $[j, m]$ -completion for a  $[j, m]$ -marked set: indeed the existence of this completion is a necessary hypothesis in almost all the characterizations of  $[j, m]$ -marked bases we gave. One may think that the definition of a  $[j, m]$ -completion is always possible starting from a  $[j, m]$ -marked set or that it is sufficient to assume that a subset  $A$  of monomials in  $j_{\leq m} \setminus B_j$  has  $[j, m]$ -reduced forms of degree  $\leq m$ . A reasonable subset of monomials to consider could be the *border basis* of  $j$  [29, Section 1]:

$$\mathcal{B}(j) = \{x_i x^\gamma \in j \mid x^\gamma \in \mathcal{N}(j)\}$$

The following example shows that this is not the case, in fact even though the monomials in  $(j_{\leq m} \setminus B_j) \cap \mathcal{B}(j)$  reduce by  $\xrightarrow{\mathfrak{G}_*}$  to polynomials of degree  $\leq m$ , however this does not imply the same property for other monomials in  $j_{\leq m} \setminus B_j$ .

**Example 4.25.** Consider  $j = (x_3, x_2^2) \subset K[x_1, x_2, x_3]$ ,  $m = 3$  and the  $[j, m]$ -marked set  $\mathfrak{G} = \{f_1, f_2\}$  with

$$f_1 = x_3 - x_1 x_2 + x_1^2 - x_2 + 2x_1 - 3, \quad f_2 = x_2^2 - x_1 x_2 - x_2 - 1,$$

with  $\text{Ht}(f_1) = x_3$ ,  $\text{Ht}(f_2) = x_2^2$ ,  $\text{supp}(T(f_i)) \subseteq \mathcal{N}(j)_{\leq 3} = \{1, x_2, x_1, x_1 x_2, x_1^2, x_1^3\}$ ,  $i = 1, 2$ . The  $[j, m]$ -marked set  $\mathfrak{G}$  fullfills condition (i) of Theorem 4.23.

For what concerns condition (ii), instead of considering the whole set

$$j_{\leq 3} \setminus B_j = \{x_3^2, x_2 x_3, x_1 x_3, x_3^3, x_3^2 x_2, x_3^2 x_1, x_3 x_2^2, x_3 x_2 x_1, x_3 x_1^2, x_2^2 x_1, x_1^3\}.$$

we just consider its subset  $A$  containing monomials which lie on the border of  $j$ :

$$A := (j_{\leq 3} \setminus B_j) \cap \mathcal{B}(j) = \{x_1 x_2^2, x_1 x_3, x_1^2 x_3, x_2 x_3, x_1 x_2 x_3\}.$$

The  $[j, m]$ -marked set  $\mathfrak{G}$  also fullfills condition (ii) of Theorem 4.23 for what concerns the monomials in  $A$ . However the reduction by  $\xrightarrow{\mathfrak{G}_*}$  of the monomial  $x_1 x_3^2 \in j_{\leq 3} \setminus B_j$  has degree strictly larger than  $m = 3$ , hence condition (ii) of Theorem 4.23 is not satisfied.

**4.4. Minimal  $m$  for a marked basis and the 0-dimensional case.** In the present subsection we will focus our attention on two issues that allow faster explicit computations: first, we will show that under suitable hypothesis on  $m$  and the monomials of degree  $m + 1$  in  $B_j$ , a  $[j, m]$ -marked basis is actually a  $[j, m - 1]$ -marked basis; secondly, we will consider the case of 0-dimensional ideal and improve the criterion of Theorem 4.23 in this special case. The first result is analogous to [5, Theorem 5.7], but the proof given here for a  $[j, m]$ -marked basis is much easier and gives a better insight in the algebraic structure of marked bases. The second result has been observed by the authors of [5] after computing several examples of marked schemes: using the “affine” constructions and tools that we have developed till here, the proof is immediate.

We consider a strongly stable ideal  $j \subseteq R$ . We now show that, under some hypothesis on the monomials of  $B_j$  of degree  $m + 1$ , if we have a  $[j, m]$ -marked basis  $\mathfrak{G}$ , actually the  $[j, m]$ -normal form of every monomial in  $R$  has degree bounded by the maximum between  $m - 1$  and the degree of the monomial itself.

**Lemma 4.26.** *Let  $j$  be a strongly stable ideal in  $R$ ,  $m$  be a positive integer such that no monomial in  $B_j$  of degree  $m + 1$  is divided by  $x_1$ . Consider a  $[j, m]$ -marked basis  $\mathfrak{G}$ . Then for every  $x^\beta \in R$ ,  $\text{supp}(\text{Nf}(x^\beta)) \subseteq \mathcal{N}(j)_{\leq t}$ , with  $t = \max\{m - 1, |\beta|\}$ .*

*Proof.* First we recall that, since  $\mathfrak{G}$  is a  $[j, m]$ -marked basis,  $[j, m]$ -reduced forms modulo  $(\mathfrak{G})$  are unique and they can be computed by  $\xrightarrow{\mathfrak{G}_*}$  (Proposition 4.12). Then for every  $x^\beta \in R$ ,  $x^\beta \xrightarrow{\mathfrak{G}_*} \text{Nf}(x^\beta)$ .

If  $x^\beta \in R \setminus j$ , then  $\text{Nf}(x^\beta) = x^\beta \in \mathcal{N}(j)_{\leq |\beta|}$ . If  $x^\beta \in j$  and  $|\beta| > m$ , then  $t = |\beta|$  and, by definition of  $[j, m]$ -reduced form modulo  $(\mathfrak{G})$ , we have  $\text{supp}(\text{Nf}(x^\beta)) \subseteq \mathcal{N}(j)_{\leq |\beta|}$ . If  $x^\beta \in j$  with  $|\beta| \leq m - 1$ , then its  $[j, m]$ -normal form has degree lower than or equal to  $m$ . Let  $x^\epsilon$  be a monomial in  $\text{supp}(\text{Nf}(x^\beta))$  with  $|\epsilon| = m$  and consider  $x_1 x^\beta$ : by Proposition 4.12, we have that  $\text{Nf}(x_1 \text{Nf}(x^\beta)) = \text{Nf}(x_1 x^\beta)$ . In particular,  $\text{supp}(\text{Nf}(x_1 x^\beta)) \subset \mathcal{N}(j)_{\leq m}$  hence  $x_1 x^\epsilon$  does not appear in  $\text{Nf}(x_1 x^\beta)$ , because  $1 + |\epsilon| = m + 1$ , and  $x_1 x^\epsilon$  is reducible by  $\xrightarrow{\mathfrak{G}_*}$ .

Consider  $x_1 x^\epsilon = x^\alpha *_j x^{\delta'}$ : by Lemma 2.3, we obtain  $x^{\delta'} <_{\text{Lex}} x_1$ , that means  $x^{\delta'} = 1$ . But this means that  $x_1$  divides  $x^\alpha \in B_j$ ,  $|\alpha| = m + 1$ , and this contradicts the hypothesis.  $\square$

For a strongly stable ideal  $j \subseteq R$ , let  $\rho$  be its satiety (Definition 3.1), which is the maximal degree of a monomial in  $B_j$  divided by  $x_1$  (Lemma 3.2).

**Theorem 4.27.** *Let  $j$  be a strongly stable ideal in  $R$ ,  $m$  be a positive integer such that no monomial in  $B_j$  of degree  $m + 1$  is divided by  $x_1$ . Consider a  $[j, m]$ -marked set of polynomials  $\mathfrak{G}$ .  $\mathfrak{G}$  is a  $[j, m - 1]$ -marked basis if and only if  $\mathfrak{G}$  is a  $[j, m]$ -marked basis. In particular, for every  $m \geq \rho$ ,  $\mathfrak{G}$  is a  $[j, m]$ -marked basis if and only if it is a  $[j, \rho - 1]$ -marked basis.*

*Proof.* If  $\mathfrak{G}$  is a  $[j, m - 1]$ -marked basis, then  $R_{\leq t} = \langle \mathcal{N}(j)_{\leq t} \rangle \oplus (\mathfrak{G})_{\leq t}$  for every  $t \geq m - 1$ . Then the same holds for every  $t \geq m \geq m - 1$ , so  $\mathfrak{G}$  is also a  $[j, m]$ -marked basis.

Vice versa, if  $\mathfrak{G}$  is a  $[j, m]$ -marked basis then  $R_{\leq t} = \langle \mathcal{N}(j)_{\leq t} \rangle \oplus (\mathfrak{G})_{\leq t}$  for every  $t \geq m$ . It is sufficient to prove that  $\mathfrak{G}$  is a  $[j, m - 1]$ -marked set and that  $R_{\leq m-1} = \langle \mathcal{N}(j)_{\leq m-1} \rangle \oplus (\mathfrak{G})_{\leq m-1}$ .

Since  $\mathfrak{G}$  is a  $[j, m]$ -marked basis, reduced forms are unique. In particular for every  $x^\alpha \in B_j$ , there is a marked polynomial  $f_\alpha \in \mathfrak{G}$  such that  $f_\alpha = x^\alpha - \text{Nf}(x^\alpha)$ . By Lemma 4.26, for every  $x^\alpha \in B_j$ ,  $|\alpha| \leq m - 1$ ,  $\deg \text{Nf}(x^\alpha) \leq m - 1$ , hence  $\mathfrak{G}$  is a  $[j, m - 1]$ -marked set.

We consider the basis of monomials for the  $K$ -vector space  $R_{\leq m-1}$ . We will prove that for every  $x^\gamma \in R$ ,  $|\gamma| \leq m - 1$ , there is a unique decomposition  $x^\gamma = g_1 + g_2$  with  $g_1 \in (\mathfrak{G})_{\leq m-1}$  and  $g_2 \in \langle \mathcal{N}(j)_{\leq m-1} \rangle$ .

If  $x^\gamma \notin j$ , then we consider the following unique writing of  $x^\gamma \in (\mathfrak{G})_{\leq m-1} \oplus \langle \mathcal{N}(j)_{\leq m-1} \rangle$ :  $x^\gamma = 0 + x^\gamma$ . If there was another such writing for  $x^\gamma$  in  $(\mathfrak{G})_{\leq m-1} + \langle \mathcal{N}(j)_{\leq m-1} \rangle$  this would be

in contradiction with the assumption that  $\mathfrak{G}$  is a  $[j, m]$ -marked basis. Then every monomial in  $R_{\leq m-1} \setminus j_{\leq m-1}$  has a unique decomposition in  $(\mathfrak{G})_{\leq m-1} + \langle \mathcal{N}(j)_{\leq m-1} \rangle$ .

We now consider  $x^\gamma \in j_{\leq m-1}$ . Since  $\mathfrak{G}$  is a  $[j, m]$ -marked basis, there is a completion  $\overline{\mathfrak{G}}$  and  $[j, m]$ -normal forms are unique, by Theorem 4.10. We can compute the unique  $[j, m]$ -reduced form modulo  $(\mathfrak{G})$  of  $x^\gamma$  (using  $\xrightarrow{\mathfrak{G}_*}$ , by Proposition 4.12). We are in the hypothesis of Lemma 4.26, hence  $\deg \text{Nf}(x^\gamma) \leq m-1$ . Hence there exists  $f_\gamma := x^\gamma - \text{Nf}(x^\gamma) \in \mathfrak{G} \cup \overline{\mathfrak{G}}$ . Then we can consider the writing  $x^\gamma = f_\gamma + \text{Nf}(x^\gamma)$ ,  $f_\gamma \in (\mathfrak{G})_{\leq m-1}$  and  $\text{Nf}(x^\gamma) \in \langle \mathcal{N}(j)_{\leq m-1} \rangle$ . If there was another such writing for  $x^\gamma \in j_{\leq m-1}$ , this would be in contradiction with the hypothesis that  $\mathfrak{G}$  is a  $[j, m]$ -marked basis.  $\square$

We obtain the following result by Theorems 4.23 and 4.27.

**Corollary 4.28.** *Let  $j$  be a strongly stable ideal in  $R$ ,  $m$  be a positive integer such that no monomial in  $B_j$  of degree  $m+1$  is divided by  $x_1$ . Let  $\mathfrak{G}$  be a  $[j, m]$ -marked set. Then  $\mathfrak{G}$  is a  $[j, m-1]$ -marked basis if and only if*

- (i) *for every  $f_\alpha \in \mathfrak{G}$ , for every  $x_i > \min(x^\alpha)$ ,  $x_i f_\alpha \xrightarrow{\mathfrak{G}_*} 0$ ;*
- (ii) *for all  $x^\beta \in j_{\leq m-1} \setminus B_j$ ,  $x^\beta \xrightarrow{\mathfrak{G}_*} g_\beta$  with  $\text{supp}(g_\beta) \subseteq \mathcal{N}(j)_{\leq m-1}$ .*

We now turn to the special case of a  $[j, m]$ -marked set  $\mathfrak{G}$  which defines an Artinian ideal. The algorithmic techniques that we can use to check whether  $\mathfrak{G}$  is a  $[j, m]$ -marked basis are more efficient in this special case.

**Proposition 4.29.** *Let  $j$  be a strongly stable ideal in  $R$  and  $\mathfrak{G}$  be a  $[j, m]$ -marked set of polynomials, for some  $m \geq \text{reg}(j) - 1$ . If  $\mathcal{N}(j)$  is a finite set, then for every  $x^\beta \in j$ ,  $x^\beta \xrightarrow{\mathfrak{G}_*} g_\beta$  with  $\text{supp}(g_\beta) \subseteq \mathcal{N}(j)_{\leq \text{reg}(j)-1}$ .*

*Proof.* For every  $x^\beta \in j$ , we compute  $x^\beta \xrightarrow{\mathfrak{G}_*} g_\beta$ ,  $\text{supp}(g_\beta) \subseteq \mathcal{N}(j)$ . We write this polynomial as  $g_\beta = g_{\beta_1} + g_{\beta_2}$ , where  $\text{supp}(g_{\beta_1}) \subseteq \mathcal{N}(j)_{\leq m}$  and  $\text{supp}(g_{\beta_2}) \subseteq \mathcal{N}(j)_{\leq t} \setminus \mathcal{N}(j)_{\leq m}$  for some  $t \geq m+1$ . Since  $\mathcal{N}(j)$  is a finite set of monomials and  $\text{reg}(j) = \max\{|\alpha| : x^\alpha \in B_j\}$ , we have that  $\mathcal{N}(j)_{\leq \text{reg}(j)-1} = \mathcal{N}(j)_{\leq \text{reg}(j)-1+r}$  for every positive integer  $r$ . Then  $g_{\beta_2} = 0$  and  $\text{supp}(g_\beta) \subseteq \mathcal{N}(j)_{\leq \text{reg}(j)-1}$ .  $\square$

**Theorem 4.30.** *Let  $j$  be a strongly stable ideal in  $R$  and  $\mathfrak{G}$  be a  $[j, m]$ -marked set of polynomials, for some positive integer  $m \geq \rho$ . If  $\mathcal{N}(j)$  is finite, then  $\mathfrak{G}$  is a  $[j, \rho-1]$ -marked basis if and only if for every  $f_\alpha \in \mathfrak{G}$  and for every  $x_i > \min(x^\alpha)$ , we have  $x_i f_\alpha \xrightarrow{\mathfrak{G}_*} 0$ .*

*Proof.* Since  $\mathcal{N}(j)$  is finite, we have that  $\rho = \text{reg}(j)$  [3, Lemma (1.7)]. Then for every  $x^\beta \in j_{\leq m-1} \setminus B_j$ , condition (ii) of Theorem 4.23 is fulfilled by Proposition 4.29.  $\square$

## 5. THE SCHEME STRUCTURE OF $\mathcal{Mf}(j, m)$ AND A FLAT FAMILY OF AFFINE SCHEMES

As recalled in Section 2, in [12] the authors consider homogeneous ideals of  $S$  generated by a  $J$ -marked basis, with  $J$  strongly stable. These ideals form a family called a  $J$ -marked family  $\mathcal{Mf}(J)$ , that is endowed with a structure of subscheme of a suitable affine space, explicitly computed by an algorithmic method based on the already cited reduction procedure  $\xrightarrow{V_\ell}$ .

In [5] it is shown that, when  $J$  is a saturated strongly stable ideal and  $m$  a positive integer, the family  $\mathcal{Mf}(J_{\geq m})$  can be embedded as a subscheme of an affine space of lower dimension than the previous one, by the superminimal reduction (Definition 2.8). Further, as shown in [5, Theorem 5.7, (i)],  $\mathcal{Mf}(J_{\geq m})$  can be embedded as a locally closed subscheme in the Hilbert scheme parameterizing the subschemes of  $\mathbb{P}^n$  having the same Hilbert polynomial as  $S/J$ . In particular, in [6] the authors highlight that  $\mathcal{Mf}(J)$  is embedded as an open subset of a Hilbert scheme when  $m$  is higher than or equal to the satiety of  $J^a \subset R$  minus one.

In this section we make analogous considerations for ideals of  $R$  generated by a  $[j, m]$ -marked basis, where  $m$  is a positive integer.

**Definition 5.1.** Let  $j$  be a strongly stable ideal in  $R$ . The family of all the ideals generated by a  $[j, m]$ -marked basis is called a  $[j, m]$ -marked family and denoted by  $\mathcal{Mf}(j, m)$ .

We consider  $j \in R$  a saturated strongly stable ideal,  $m$  a positive integer and we define the following  $[j, m]$ -marked set  $\mathfrak{G}$ :

$$(5.1) \quad \mathfrak{G} = \{f_\alpha = x^\alpha - \sum C_{\alpha\gamma} x^\gamma : Ht(f_\alpha) = x^\alpha \in B_j\} \subset K[C][x_1, \dots, x_n],$$

where  $C$  is a compact notation for the set of new variables  $C_{\alpha\gamma}$ ,  $x^\alpha \in B_j$ ,  $x^\gamma \in \mathcal{N}(j)_{\leq \max\{|\alpha|, m\}}$ .

As shown for the families considered in [12, 5], here we show that also  $\mathcal{Mf}(j, m)$  can be endowed with a structure of affine scheme, by the results of the previous sections.

**Definition 5.2.** Let  $j$  be a strongly stable ideal in  $R$ ,  $m$  be a positive integer and  $\mathfrak{G}$  be a  $[j, m]$ -marked set as in (5.1).

For every  $x^\beta \in j_{\leq m} \setminus B_j$ , let  $g_\beta$  the reduced polynomial such that  $x^\beta \xrightarrow{\mathfrak{G}*} g_\beta$ ; let  $g'_\beta$ ,  $g''_\beta$  be reduced polynomials such that  $g_\beta = g'_\beta + g''_\beta$  and  $\text{supp}(g'_\beta) \subseteq \mathcal{N}(j)_{\leq m}$  and  $\text{supp}(g''_\beta) \subseteq \mathcal{N}(j) \setminus \mathcal{N}(j)_{\leq m}$ .

We denote by

- $\mathcal{D}_1 \subset K[C]$  the set containing the coefficients of  $g''_\beta$ , for every  $x^\beta \in j_{\leq m} \setminus B_j$ ;
- $\mathcal{D}_2 \subset K[C]$  the set containing the coefficients of all the reduced polynomials in  $(\mathfrak{G})K[C][x_1, \dots, x_n]$ .

Let  $\mathfrak{A}$  be the ideal in  $K[C]$  generated by  $\mathcal{D}_1 \cup \mathcal{D}_2$ .

**Theorem 5.3.** Let  $j$  be a strongly stable ideal in  $R$ ,  $m$  be a positive integer. Then  $\mathcal{Mf}(j, m)$  is endowed with the structure of affine scheme, defined by the ideal  $\mathfrak{A} \subset K[C]$  of Definition 5.2, and  $\mathcal{Mf}(j, m) = \mathcal{Mf}(j^h_{\geq m})$  scheme-theoretically.

*Proof.* It is enough to recall that there is an equivalence (up to powers of  $x_0$ ) between marked bases in  $S$  and  $R$  (Theorem 4.18), so  $\mathcal{Mf}(j, m)$  and  $\mathcal{Mf}(j^h_{\geq m})$  are the same sets.

Further, the ideal  $\mathfrak{A}$  of Definition 5.2 is exactly the ideal defining the scheme structure of  $\mathcal{Mf}(j^h_{\geq m})$  [5, Theorem 5.4].  $\square$

**Corollary 5.4.** Let  $J$  be a saturated strongly stable ideal in  $S$ ,  $m$  be a positive integer, then  $\mathcal{Mf}(J_{\geq m}) = \mathcal{Mf}(J^a, m)$  scheme-theoretically.

**Remark 5.5.** If  $j \subset R$  is a strongly stable ideal,  $m$  is a positive integer and  $\mathfrak{G}$  is a  $[j, m]$ -marked set, the ideal  $(\mathfrak{G}) \subseteq R$  belongs to  $\mathcal{Mf}(j, m)$  if and only if  $\mathfrak{G}$  is obtained from (5.1) replacing the variables  $C$  by  $c \in K^{[C]}$  belonging to  $V(\mathfrak{A})$ . Indeed, it is sufficient to observe the following: a  $[j, m]$ -marked set  $\mathfrak{G} \subset R$ , with a completion  $\overline{\mathfrak{G}}$ , generates an ideal belonging to  $\mathcal{Mf}(j, m)$  if and only if the coefficients of the polynomials in the  $j^h_{\geq m}$ -marked set  $G$  (constructed as in Lemma 4.16) constitute a point of the scheme  $\mathcal{Mf}(j^h_{\geq m})$ .

**Proposition 5.6.** Let  $j$  be a strongly stable ideal in  $R$ ,  $m$  be a positive integer and  $\mathfrak{G}$  be a  $[j, m]$ -marked set as in (5.1). For every  $f_\alpha \in \mathfrak{G}$ , for every  $x_i > \min(x^\alpha)$ , let  $h_{\alpha,i}$  the reduced polynomial such that  $x_i f_\alpha \xrightarrow{\mathfrak{G}*} h_{\alpha,i}$  and let  $\mathcal{D}'_2 \subset K[C]$  be the set containing the coefficients of all the polynomials  $h_{\alpha,i}$ .

Then the ideal generated by  $\mathcal{D}_1 \cup \mathcal{D}'_2$  in  $K[C]$  is  $\mathfrak{A}$ .

*Proof.* Thanks to Theorems 5.3, 4.20 and 4.19, we have that the ideal generated by  $\mathcal{D}_1 \cup \mathcal{D}'_2$  in  $K[C]$  satisfies the conditions of [5, Proposition 5.5], hence  $(\mathcal{D}_1 \cup \mathcal{D}'_2) = \mathfrak{A}$ .  $\square$

**Remark 5.7.** In Section 6 we will improve the computational techniques presented in [5]: first, we can use the results of Section 4.4 to design an algorithm computing a set of generators for the ideal  $\mathfrak{A}$  which takes into account some features of the strongly stable ideal in order to get a faster execution; furthermore, although the reduction  $\xrightarrow{\mathfrak{G}_*}$  is theoretically analogous to the “superminimal reduction” defined and used in [5] by Theorems 4.20 and 4.19, from the computational viewpoint the relation  $\xrightarrow{\mathfrak{G}_*}$  allows us to avoid the time-consuming operation of multiplying by powers of  $x_0$  the polynomials we want to reduce.

The algorithm we present in Section 6 relies on Proposition 5.6, which requires a finite number of computations to construct the set  $\mathcal{D}'_2$ , while it is not possible to compute the set  $\mathcal{D}_2$  in a finite number of steps. However, in the near future, we aim to design a performing algorithm computing a subset  $\mathcal{M}$  of  $K[C]$  such that  $\mathfrak{A} \supset \mathcal{M}$  and  $(\mathcal{M}) \supset \mathfrak{A}$  in  $K[C]$ : such a set  $\mathcal{M}$  will not necessarily be computed by  $\xrightarrow{\mathfrak{G}_*}$ .

Given a strongly stable ideal  $J$  in  $S$ , [12, Proposition 4.6] shows that the marked scheme  $\text{Mf}(J)$  is flat (as defined in [22, Section 9, Chapter III]) at the origin (which corresponds to  $J$ ). We now give an independent proof that the analogous result holds for  $\text{Mf}(\mathfrak{j}, m)$ .

**Theorem 5.8.** *Let  $\mathfrak{j}$  be a strongly stable ideal in  $R$ ,  $m$  a positive integer and  $\mathfrak{G}$  be a  $[\mathfrak{j}, m]$ -marked set as in (5.1). Then  $\text{Mf}(\mathfrak{j}, m)$  is flat at the origin.*

*Proof.* First, we observe that the family  $\text{Mf}(\mathfrak{j}, m)$  is defined by the morphism

$$\text{Spec}(K[C][x_1, \dots, x_n]/(\mathfrak{A}, \mathfrak{G})) \rightarrow \text{Spec}(K[C]/\mathfrak{A}).$$

We consider  $A := \left( \frac{K[C]}{\mathfrak{A}} \right)_{(C)/\mathfrak{A}}$ ,  $\mathcal{O} := K[x_1, \dots, x_n]/\mathfrak{j}$  and  $\mathcal{O}_A := A[X]/(\mathfrak{G})$ . By Corollary 4.24, we obtain that every syzygy of  $J$  lifts to a syzygy of the ideal generated by  $\mathfrak{G}$  in  $\mathcal{O}_A$ . Hence, by [1, Corollary to Proposition 3.1, Chapter 1],  $\mathcal{O}_A$  is flat over  $A$ . Letting  $P := (C, X)/(\mathfrak{A}, \mathfrak{G})$ , in particular we obtain that  $(\mathcal{O}_A)_{(P)}$  is flat over  $A$ , hence  $\text{Mf}(\mathfrak{j}, m)$  is flat at the origin.  $\square$

In general, a family of ideals in  $R$  is not *flat*, even assuming that the ideals in the family share the same Hilbert polynomial, as shown by the following example.

**Example 5.9.** Consider  $R = \mathbb{C}[x_1, \dots, x_5]$  and the term order  $\prec$  associated to the weight vector  $\omega = [8, 7, 5, 4, 3]$ :  $x^\alpha \prec x^\beta$  if and only if  $\alpha \cdot \omega \leq \beta \cdot \omega$  and, if equality holds, ties are broken using reverse lexicographic order.

Let  $\mathfrak{U}_T$  be the set of the following polynomials:

$$\begin{aligned} \mathfrak{f}_1 &:= -212x_2x_3 - 317x_2x_4 + 284x_3x_4 + 72x_2^2 + 144x_4^2 - 43x_1x_2, \\ \mathfrak{f}_2 &:= x_1x_4 - x_3x_4 - x_1x_2 + x_2x_3, \quad \mathfrak{f}_3 := -2x_2x_3 - x_2x_4 + x_3x_4 + x_1x_3 + x_1x_2, \\ \mathfrak{f}_4 &:= 4x_2x_3 + 4x_0x_4 + x_2x_4 - 4x_3x_4 - 5x_1x_2, \quad \mathfrak{f}_5 := -5x_2x_4 + 4x_3x_4 + 4x_0x_3 - 8x_2x_3 + 5x_1x_2, \\ \mathfrak{f}_6 &:= -4x_3x_4 + 4x_0x_2 + 4x_2x_3 + 3x_2x_4 - 7x_1x_2, \\ \mathfrak{f}_7 &:= 71x_2x_4 - 90x_4^2 + 18x_1^2 - 56x_3x_4 + 18x_3^2 - 34x_2x_3 + 109x_1x_2, \\ \mathfrak{f}_8 &:= -5x_2x_4 + 4x_3x_4 + 4x_0x_1 - 4x_2x_3 + x_1x_2, \\ \mathfrak{f}_9 &:= -83x_2x_4 + 48x_4^2 + 68x_3x_4 + 12x_0^2 - 44x_2x_3 - 25x_1x_2, \\ \mathfrak{f}_{10}(T) &:= x_1x_2 - x_2x_3 + T(x_4^3 + x_3^3 + x_2^3). \end{aligned}$$

For every  $\tau \in \mathbb{C}$  all the ideals  $\mathfrak{i}_\tau \subset \mathbb{C}[x_1, \dots, x_5]$ , generated by the set of polynomials  $\mathfrak{U}_\tau$  obtained specializing  $T$  to  $\tau$ , share the same affine Hilbert polynomial  $p(t) = 12$ . Moreover the following monomial ideal is the initial ideal w.r.t.  $\prec$  of every ideal  $\mathfrak{i}_\tau$ :

$$\mathfrak{j} = (x_5^2, x_5x_4, x_4^2, x_5x_3, x_4x_2, x_3^2, x_5x_2, x_4x_2, x_5x_1, x_4x_1, x_3x_2^2, x_2^3, x_3x_2x_1, x_3^2x_1, x_3x_1^2, x_2x_1^2, x_1^4).$$

In fact,  $\mathfrak{j}$  is the initial ideal of the ideal  $\mathfrak{J}$  generated by  $\mathfrak{U}_T$  in  $\mathbb{C}[T, \frac{1}{T}][x_1, \dots, x_5]$ ; all the polynomials in its reduced Gröbner basis have coefficients in  $\mathbb{C}[T]$ , except the one with leading



monomial  $x_1^4$ , which is  $\mathfrak{g}_T = x_1^4 + \frac{4T-1}{9T}x_1^3$ . Thus, specializing  $T \mapsto \tau \in \mathbb{C} \setminus \{0\}$ , we get the reduced Gröbner basis  $B_\tau$  of the ideal  $\mathfrak{i}_\tau$ , hence the initial ideal of  $\mathfrak{i}_\tau$  w.r.t.  $\prec$  is  $\mathfrak{j}$ .

Finally, if  $\tau = 0$ , by a direct computation of the reduced Gröbner basis  $B$  with respect to  $\prec$  of  $\mathfrak{i}_0$ , we obtain that its initial ideal is  $\mathfrak{j}$  too. Observe that  $B$  contains the polynomial  $x_1^4$  and does not contain the monomial  $x_1^3$  as one could expect, observing that  $-9T\mathfrak{g}_T = -9Tx_1^4 + (-4T+1)x_1^3 \in \mathfrak{J}$ .

We underline that  $\mathfrak{j}$  is a strongly stable ideal and also an affine 3-segment with respect to  $\omega = [8, 7, 5, 4, 3]$  (Definition 7.1). Hence, for every  $\tau \in \mathbb{C}$ , both  $\tau \neq 0$  and  $\tau = 0$ , the reduced Gröbner basis w.r.t.  $\prec$  of  $\mathfrak{i}_\tau$  is a  $[\mathfrak{j}, 3]$ -marked basis, but the marked basis for  $\mathfrak{i}_0$  is not the limit of those of  $\mathfrak{i}_\tau$  with  $\tau \neq 0$ .

We observe that, for  $\tau = 0$ ,  $\text{Proj}(S/(\mathfrak{i}_0)^h)$  is a Gorenstein scheme in  $\mathbb{P}^5$ , with Hilbert polynomial 12, while, for  $\tau \neq 0$ ,  $\text{Proj}(S/(\mathfrak{i}_\tau)^h)$  is not Gorenstein. Since Gorenstein schemes constitute an open subset of  $\mathcal{H}\text{ilb}_{12}^5$ , this means that the family of ideals in  $R$  given by  $\mathfrak{i}_\tau$  is not flat. Furthermore, if we homogenize the polynomials in  $B_\prec$  and then we replace  $T$  by 0, we do not get the ideal  $(\mathfrak{i}_0)^h$ .

Summing up  $\mathfrak{i}_\tau$  is a family whose ideals have constant affine Hilbert Polynomial 12, constant initial ideal  $\mathfrak{j}$  with respect to  $\prec$ , are generated by a marked basis over the strongly stable ideal  $\mathfrak{j}$ , but this family is not flat. In fact, our family defines a function  $\mathbb{A}^1 \rightarrow \mathcal{M}\text{f}(\mathfrak{j}, 3)$  which is not a morphism of schemes.

We end this section by presenting two interesting features of  $\mathcal{M}\text{f}(\mathfrak{j}, m)$ , which are both closely connected to the flatness at  $\mathfrak{j}$ .

In the following proposition we show how to obtain simultaneously the homogenization of all the ideals belonging to a  $[\mathfrak{j}, m]$ -marked family.

**Proposition 5.10.** *Let  $\mathfrak{j}$  be a strongly stable ideal in  $R$ ,  $m$  be a positive integer,  $\mathfrak{G}$  be the  $[\mathfrak{j}, m]$ -marked set in  $K[C][x_1, \dots, x_n]$  as in (5.1). Let  $\mathfrak{A}$  be the ideal in  $K[C]$  which defines the affine scheme structure of  $\mathcal{M}\text{f}(\mathfrak{j}, m)$  and let  $\overline{\mathfrak{G}}$  be the completion of  $\mathfrak{G}$  containing, for every  $x^\beta \in (\mathfrak{j})_{\leq m} \setminus B_{\mathfrak{j}}$ , the polynomials  $x^\beta - h_\beta \pmod{\mathfrak{A}}$ , where  $x^\beta \xrightarrow{\mathfrak{G}_*} h_\beta \pmod{\mathfrak{A}}$ .*

*Then  $(\mathfrak{G}^h \cup \overline{\mathfrak{G}}^h)_{\geq m}$  is an  $m$ -saturated ideal and it is equal to  $(\mathfrak{G})_{\geq m}^h \subseteq K[C, x_0, \dots, x_n]/\mathfrak{A}$ , having  $\mathfrak{j}_{\geq m}^h$ -marked basis  $G$ , the set of polynomials constructed as in Lemma 4.16.*

*Proof.* It is sufficient to apply Proposition 4.21 to the  $[\mathfrak{j}, m]$ -marked basis  $\mathfrak{G} \subset K[C, x_1, \dots, x_n]/\mathfrak{A}$  and to its completion.  $\square$

Let  $K$  be a field of characteristic 0. With the same notations already fixed in Section 2, let  $p(t)$  be an *admissible Hilbert polynomial* in  $S$ . The *Hilbert scheme*  $\mathcal{H}\text{ilb}_{p(t)}^n$  is the projective scheme parameterizing the subschemes  $\mathcal{Z}$  of  $\mathbb{P}^n$  having Hilbert polynomial  $p(t)$ . If  $I$  is a homogeneous ideal defining the scheme  $\mathcal{Z}$ , we will say that  $I$  is a point of  $\mathcal{H}\text{ilb}_{p(t)}^n$ .

**Theorem 5.11.** *Let  $\mathfrak{j}$  be a strongly stable ideal in  $R$ ,  $m$  be a positive integer, and assume that  $m \geq \text{sat}(\mathfrak{j}) - 1$ . Then  $\mathcal{M}\text{f}(\mathfrak{j}, m)$  is isomorphic to an open subset of the Hilbert scheme  $\mathcal{H}\text{ilb}_{p(t)}^n$ , where  $p(t)$  is the Hilbert polynomial of  $S/\mathfrak{j}^h$ . Furthermore, if  $\mathcal{S}_{p(t)}^n$  is the set of strongly stable ideals in  $R$  with affine Hilbert polynomial  $p(t)$ , then, up to linear changes of coordinates in  $\mathbb{P}^n$ , the open subsets  $\mathcal{M}\text{f}(\mathfrak{j}, \text{sat}(\mathfrak{j}) - 1)$  cover  $\mathcal{H}\text{ilb}_{p(t)}^n$ , for  $\mathfrak{j} \in \mathcal{S}_{p(t)}^n$ .*

*Proof.* By using the scheme-theoretical equality between  $\mathcal{M}\text{f}(\mathfrak{j}, m)$  and  $\mathcal{M}\text{f}(\mathfrak{j}_{\geq m}^h)$  established in Theorem 5.3, we get that  $\mathcal{M}\text{f}(\mathfrak{j}, m)$  is isomorphic to an open subset of  $\mathcal{H}\text{ilb}_{p(t)}^n$  for every  $m \geq \text{sat}(\mathfrak{j}) - 1$  by [6, Theorem 3.1] and these open subsets cover  $\mathcal{H}\text{ilb}_{p(t)}^n$  up to changes of coordinates in  $\mathbb{P}^n$  applying [6, Theorem 2.5].  $\square$

## 6. ALGORITHMS

We now describe a prototype of the algorithm for computing  $[j, m]$ -marked families based on Theorem 5.3 and Proposition 5.6, with a variant in case the strongly stable ideal  $j$  has a finite sous-escalier, as observed in Proposition 4.29 and Theorem 4.30.

Let us suppose that the following functions are made available. The ideal  $j \subseteq R$  is always strongly stable.

- **AFF**( $J$ ). It returns the monomial ideal  $J^a \subseteq R$ , given a monomial ideal  $J \subseteq S$ .
- **COEFF**( $\mathfrak{g}, x^\gamma$ ). It returns the coefficient of the monomial  $x^\gamma$  in the polynomial  $\mathfrak{g}$  (obviously 0 if  $x^\gamma \notin \text{supp}(\mathfrak{g})$ ).
- **BASIS**( $j$ ). It determines the minimal set of monomials generating  $j$ .
- **LOWERPART**( $\mathfrak{g}, t$ ). Given a polynomial  $\mathfrak{g}$  and a non-negative integer  $t$ , it returns the pair of polynomials  $(\mathfrak{g}_1, \mathfrak{g}_2)$  such that  $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ ,  $\deg(\mathfrak{g}_1) \leq t$  and  $\text{supp}(\mathfrak{g}_2) \subset R_{\geq t+1}$ .
- **OPTIMIZEDLEVEL**( $j, m$ ). It determines the biggest integer  $m_0 \leq m$  such that there is  $x^\alpha \in B_j$ ,  $|\alpha| = m_0 + 1$  such that  $x_1$  divides  $x^\alpha$ .
- **REG**( $j$ ). It determines the regularity of  $j$ , using Lemma 3.2.
- **SAT**( $j$ ). It determines the satiety of  $j$ , using Lemma 3.2.
- **SOUS-ESCALIER**( $j, m$ ). It determines the set of monomials in the sous-escalier of  $j$ , up to degree  $m$ .
- **SUPERMINIMALREDUCTION**( $\mathfrak{g}, \mathfrak{G}$ ). Given a  $[j, m]$ -marked set  $\mathfrak{G}$  and a polynomial  $\mathfrak{g}$ , it returns the polynomial  $\mathfrak{g}_1$  such that  $\mathfrak{g} \xrightarrow{\mathfrak{G}_*} \mathfrak{g}_1$ ,  $\text{supp}(\mathfrak{g}_1) \subset \mathcal{N}(j)$  (according to Definition 4.3 and Theorem 4.4).
- **VSPACE**( $j, m$ ). It determines the monomial basis of the  $K$ -vector space  $j_{\leq m}$ .

```

1: REDUCTION1( $j, m, G$ )
Input:  $j \subset K[x_1, \dots, x_n]$  strongly stable ideal,  $m$  a positive integer,  $\mathfrak{G} \subset K[x_1, \dots, x_n]$  a
 $[j, m]$ -marked set.
Output: the conditions to impose on the coefficients of the polynomials in  $\mathfrak{G}$  in order to
satisfy condition (i) of Theorem 4.23.
2: Equations1  $\leftarrow \emptyset$ ;
3: for all  $f_\alpha \in \mathfrak{G}, x_i > \min(x^\alpha)$  do
4:    $\mathfrak{g} \leftarrow \text{SUPERMINIMALREDUCTION}(x_i f_\alpha, \mathfrak{G})$ ;
5:   for all  $x^\gamma \in \text{supp}(\mathfrak{g})$  do
6:     Equations1  $\leftarrow \text{Equations1} \cup \{\text{COEFF}(\mathfrak{g}, x^\gamma)\}$ ;
7:   end for
8: end for
9: return Equations1;

```

```

1: REDUCTION2( $j, m, G$ )
Input:  $j \subset K[x_1, \dots, x_n]$  strongly stable ideal,  $m$  a positive integer,  $\mathfrak{G} \subset K[x_1, \dots, x_n]$  a
 $[j, m]$ -marked set.
Output: the conditions to impose on the coefficients of the polynomials in  $\mathfrak{G}$  in order to
satisfy condition (ii) of Theorem 4.23.
2: Equations2  $\leftarrow \emptyset$ ;
3:  $B \leftarrow \text{VSPACE}(j, m) \setminus \text{BASIS}(j)$ ;
4: for all  $x^\beta \in B$  do
5:    $\mathfrak{g} \leftarrow \text{SUPERMINIMALREDUCTION}(x^\beta, \mathfrak{G})$ ;
6:    $(\mathfrak{g}_1, \mathfrak{g}_2) \leftarrow \text{LOWERPART}(\mathfrak{g}, m)$ ;
7:   for all  $x^\gamma \in \text{supp}(\mathfrak{g}_2)$  do
8:     Equations2  $\leftarrow \text{Equations2} \cup \{\text{COEFF}(\mathfrak{g}_2, x^\gamma)\}$ ;
9:   end for
10: end for
11: return Equations2;

```

```

1: MARKEDSCHEME( $j, m$ )
Input:  $j \subset K[x_1, \dots, x_n]$  strongly stable ideal,  $m$  a positive integer.
Output: an ideal defining the marked scheme  $\mathcal{Mf}((j^h)_{\geq m})$ .
2: Equations  $\leftarrow \emptyset$ 
3:  $B \leftarrow \text{BASIS}(j)$ ;
4:  $m_0 \leftarrow \text{OPTIMIZEDLEVEL}(j, m)$ ;
5:  $\mathfrak{G} \leftarrow \emptyset$ ;
6: for all  $x^\alpha \in B$  do
7:    $\mathfrak{f}_\alpha \leftarrow x^\alpha$ ;
8:   for all  $x^\eta \in \text{SOUSDESCALIER}(j, \max\{m_0, |\alpha|\})$  do
9:      $\mathfrak{f}_\alpha \leftarrow \mathfrak{f}_\alpha + C_{\alpha\eta}x^\eta$ ;
10:  end for
11:   $\mathfrak{G} \leftarrow \mathfrak{G} \cup \{\mathfrak{f}_\alpha\}$ ;
12: end for
13: Equations  $\leftarrow \text{REDUCTION1}(j, m_0, \mathfrak{G})$ 
14: if  $m_0 = \text{REG}(j) - 1$  and  $\text{SOUSDESCALIER}(j, \text{REG}(j) - 1) = \text{SOUSDESCALIER}(j, \text{REG}(j))$  then
15:   return Equations
16: end if
17: Equations  $\leftarrow \text{Equations} \cup \text{REDUCTION2}(j, m_0, \mathfrak{G})$ 
18: return (Equations)

```

Finally, given a saturated strongly stable ideal  $J \subseteq S$ , the algorithm MARKEDSCHEME is used to compute equations for the  $J_{\geq \rho-1}$ -marked scheme, with  $\rho = \text{sat}(J^a)$ . Indeed,  $\rho - 1$  is the minimum integer ensuring that  $\mathcal{Mf}(J^a, \rho - 1) \simeq \mathcal{Mf}(J_{\geq \rho-1})$  is a open subset of  $\text{Hilb}_{p(t)}^n$ , where  $p(t)$  is the Hilbert Polynomial of  $S/J$  (see [6, Theorem 3.1] and Theorem 5.11).

```

1: HILBERTOPENSUBSET( $J$ )
Input:  $J \subset K[x_0, x_1, \dots, x_n]$  saturated strongly stable monomial ideal.
Output: a set of generators of the ideal defining the marked scheme  $\mathcal{Mf}(J_{\geq \rho-1})$ , where
 $\rho = \text{sat}(J^a)$ .
2: return MARKEDSCHEME(AFF( $J$ ), SAT(AFF( $J$ )) - 1)

```

## 7. EXPLICIT COMPUTATIONS ON A HILBERT SCHEME OF POINTS

In this section we will present a significant application of the computational method described in the previous sections.

Let  $K$  be a field of characteristic 0. We consider an admissible Hilbert polynomial  $p(t)$  and the Hilbert scheme  $\mathcal{Hilb}_{p(t)}^n$  parameterizing the subschemes  $\mathcal{Z}$  of  $\mathbb{P}^n$  having Hilbert polynomial  $p(t)$ . If  $I$  is a homogeneous ideal defining the scheme  $\mathcal{Z}$ , we will say that  $I$  is a point of  $\mathcal{Hilb}_{p(t)}^n$ .

Among saturated homogeneous ideals  $I$  such that the Hilbert polynomial of  $S/I$  is  $p(t)$ , there is a well-known one, the monomial ideal  $J_{\text{Lex}}$ , which is the **Lex**-segment, also called in this setting the **Lex-point** of  $\mathcal{Hilb}_{p(t)}^n$ . The **Lex**-point is a smooth point of  $\mathcal{Hilb}_{p(t)}^n$  (see [30]), hence it belongs to a single irreducible component of  $\mathcal{Hilb}_{p(t)}^n$ , that we denote by  $\mathcal{H}_{\text{Lex}}$ .

Even in the easiest situation, namely a Hilbert scheme parameterizing 0-dimensional schemes, it is not known yet, except in a few special cases, how many irreducible components  $\mathcal{Hilb}_{p(t)}^n$  has. Indeed, one of the main problems to overcome is the lack of computational tools allowing the direct study of  $\mathcal{Hilb}_{p(t)}^n$ . It is quite natural to embed  $\mathcal{Hilb}_{p(t)}^n$  as a closed subscheme of suitable Grassmannians [19, 10, 23] and consider equations defining this scheme structure. Different authors studied bounds for the degree of a set of generators: a recent paper [7] improves the known bounds given by [2, 21, 23], showing that there is a set of defining equations whose degree is lower than or equal to  $\deg(p(t)) + 2$ . Nevertheless, direct computations using these sets of defining equations are impossible because of the large number of variables involved.

Since the direct study of the equations describing  $\mathcal{Hilb}_{p(t)}^n$  in the Grassmannian is not affordable, it is reasonable to study it locally, by a suitable open cover. In [6] it is proved that if  $J$  is a saturated strongly stable ideal belonging to  $\mathcal{Hilb}_{p(t)}^n$ , then  $\mathcal{Mf}(J_{\geq r})$  is an open subset of  $\mathcal{Hilb}_{p(t)}^n$ , where  $r$  is the Gotzmann number of  $p(t)$ . Further, up to linear changes of coordinates in  $\mathbb{P}^n$ , as  $J$  varies among saturated strongly stable ideals of  $\mathcal{Hilb}_{p(t)}^n$ , the families  $\mathcal{Mf}(J_{\geq r})$  cover  $\mathcal{Hilb}_{p(t)}^n$ . This is not really surprising, since strongly stable ideals are well-distributed on any Hilbert scheme, in the sense that there is at least one strongly stable ideal lying on each irreducible component and on each intersection among irreducible components of the Hilbert scheme.

Thanks to Theorem 5.11, we can consider the open cover of  $\mathcal{Hilb}_{p(t)}^n$  made up of  $[j, \text{sat}(j) - 1]$ -marked families, for  $j$  strongly stable ideal in  $R$  with  $R/j$  having affine Hilbert polynomial  $p(t)$ , and apply the computational “affine” techniques developed mainly in Section 4 to study the Hilbert scheme.

We now show a result which is inferred from results in [17] and is analogous to results contained in [27] for the homogeneous case. For more details on segments, see also [11].

**Definition 7.1.** Let  $j$  be a strongly stable ideal in  $R$ ,  $m$  a positive integer. The ideal  $j$  is an *affine  $m$ -segment* if there is a weight vector  $\omega \in \mathbb{N}^n$  such that for every  $x^\alpha \in B_j$ ,  $\deg_\omega(x^\alpha) > \deg_\omega(x^\gamma)$  for every  $x^\gamma \in \mathcal{N}(j)_{\leq t}$ , with  $t = \max\{m, |\alpha|\}$ .

**Theorem 7.2.** Let  $j$  be a strongly stable ideal in  $R$ ,  $m$  be a positive integer and assume that  $j$  is an affine  $m$ -segment. Then every irreducible component  $\mathcal{M}$  of  $\mathcal{Mf}(j, m)$  contains  $j$ , hence  $\mathcal{Mf}(j, m)$  is a connected scheme.

If moreover  $j$  is smooth on  $\mathcal{M}$ , then  $\mathcal{M}$  is isomorphic to an affine space.

*Proof.* By [17, Corollary 2.7], since  $j$  is an affine  $m$ -segment, then  $\mathcal{Mf}(j, m)$  is a  $\omega$ -cone, where  $\omega$  is the weight vector of Definition 7.1. From this, the thesis follows.  $\square$

We now apply our results to the study of two questions about Hilbert scheme of points, focusing on some of its irreducible components.

### 7.1. Irreducible components of $\mathcal{H}\text{ilb}_{16}^7$ and smoothability of local Gorenstein algebras $(1, 7, 7, 1)$ .

We consider  $\mathcal{H}\text{ilb}_{16}^7$  that parameterizes 0-dimensional subschemes of  $\mathbb{P}^7$  of length 16. We can identify every point of  $\mathcal{H}\text{ilb}_{p(t)}^n$  with an ideal in  $R = K[x_1, \dots, x_7]$ , non-necessarily homogeneous: this is possible on a open dense subset of  $\mathcal{H}\text{ilb}_{p(t)}^n$ , by considering a generic change of coordinates. Hence, we only consider the polynomial ring  $R$ , with variables ordered as  $x_7 > \dots > x_1$ , and the ideals in  $R$  with affine Hilbert polynomial  $p(t) = 16$ .

The Lex-point of  $\mathcal{H}\text{ilb}_{16}^7$  is given by the strongly stable ideal in  $R$ :

$$\mathfrak{j}_{\text{Lex}} = (x_7, x_6, x_5, x_4, x_3, x_2, x_1^{16}).$$

It is a smooth point of  $\mathcal{H}_{\text{Lex}}$ , whose dimension is  $7 \cdot 16 = 112$  since its general point is the reduced scheme of 16 distinct points.

We can compute the complete set of strongly stable ideals of  $R$  lying on  $\mathcal{H}\text{ilb}_{16}^7$  by using the algorithm described in [11] and further developed and implemented in [26], obtaining 561 ideals (total time of computation: about 1 second). We focus on one of them, the number 541, denoted by  $\mathfrak{j}$  and generated by the following monomials:

$$\begin{aligned} & x_7^2, x_7x_6, x_7x_5, x_7x_4, x_7x_3, x_7x_2, x_7x_1, x_6^2, x_6x_5, x_6x_4, x_6x_3, x_6x_2, x_6x_1, x_5^2, x_5x_4, x_5x_3, x_5x_2, \\ & x_5x_1, x_4^2, x_4x_3, x_4x_2, x_4x_1^2, x_3^3, x_3^2x_2, x_3^2x_1, x_3x_2^2, x_3x_2x_1, x_3x_1^2, x_2^3, x_2^2x_1, x_2x_1^2, x_1^4. \end{aligned}$$

Our interest in this special strongly stable ideal comes from the following fact: it is the generic initial ideal w.r.t. Lex of a general ideal defining a local Gorenstein algebra of type  $(1, 7, 7, 1)$ .

By Theorem 5.11,  $\mathcal{M}\text{f}(\mathfrak{j}, 3)$  is an open subset of  $\mathcal{H}\text{ilb}_{16}^7$ . Furthermore,  $\mathfrak{j}$  is an affine  $m$ -segment with weight vector  $\omega = [11, 10, 9, 8, 6, 5, 4]$ , hence  $\mathcal{M}\text{f}(\mathfrak{j}, 3)$  is connected and all its irreducible components contain  $\mathfrak{j}$  (Theorem 7.2).

We can construct the  $[\mathfrak{j}, 3]$ -marked set  $\mathfrak{G}$  in  $K[C, x_1, \dots, x_7]$  (as in (5.1)) and apply the Algorithm MARKEDSCHEME with input  $\mathfrak{j}$  and 3. Observe that since we are considering a strongly stable ideal  $\mathfrak{j}$  with a finite sous-escalier, the Algorithm MARKEDSCHEME will only call the Algorithm REDUCTION1 and not REDUCTION2.

We get  $\mathcal{M}\text{f}(\mathfrak{j}, 3)$  as the affine scheme defined by an ideal  $\mathfrak{A}$  generated by 2160 polynomials of degrees between 3 and 5 in the polynomial ring  $K[C]$ , in 512 variables.

We would like to study the irreducible components of  $\mathcal{M}\text{f}(\mathfrak{j}, 3)$ , since their closures are irreducible components of  $\mathcal{H}\text{ilb}_{16}^7$ . The computation of a primary decomposition of  $\mathfrak{A}$  with Gröbner-like techniques is absolutely unaffordable. However we can obtain interesting information on some of the irreducible components of  $\mathcal{M}\text{f}(\mathfrak{j}, 3)$  by our computational techniques.

The irreducible component  $\mathcal{M}_1$  whose closure is  $\mathcal{H}_{\text{Lex}}$  is the one on which the Plücker coordinate corresponding to  $\mathfrak{j}_{\text{Lex}}$  does not identically vanish. It is well known that  $\mathcal{H}_{\text{Lex}}$  is rational: we obtain a rational parametrization of  $\mathcal{M}_1$  choosing a suitable subset of 112 variables  $C_1 \subset C$  as parameters, but, since  $\mathfrak{j}$  is a singular point of  $\mathcal{M}_1$ , in this way we cannot obtain a polynomial parametrization. However we can easily find parametrizations for some special families of ideals contained in  $\mathcal{M}_1$ . Let us consider the following  $[\mathfrak{j}, 3]$ -marked set  $\mathfrak{G}_t$  whose coefficients depend on a parameter  $t$ :

$$\begin{aligned} \mathfrak{g}_1 &:= \mathbf{x}_7^2 + 4x_1x_2 - 2x_1x_3 - x_2^2 + x_2x_3 - 4x_1x_4 - 2x_3^2 + (-9x_7 + 16x_4 + \frac{11}{2}x_3 - 7x_2 + 2x_1)t - 8t^2 \\ \mathfrak{g}_2 &:= \mathbf{x}_7\mathbf{x}_6 + x_1x_2 - x_1x_4 - x_2x_3 + (-\frac{1}{2}x_6 + x_4 + \frac{1}{2}x_3 - x_2 + \frac{1}{2}x_1)t - \frac{1}{2}t^2 \\ \mathfrak{g}_3 &:= \mathbf{x}_7\mathbf{x}_5 - x_1x_2 + x_1x_4 + x_1x_3 - x_7t - \frac{1}{2}tx_5 - tx_4 - tx_3 + tx_2 - \frac{1}{2}tx_1 + t^2, \\ \mathfrak{g}_4 &:= \mathbf{x}_7\mathbf{x}_4 - 2x_2x_3 - 5x_1x_2 + 2x_1x_4 + 5x_1x_3 + 2x_3^2 - x_1^2 + 11x_7t - \frac{45}{2}tx_4 - 8tx_3 + 5tx_2 - \frac{3}{2}tx_1 + 17t^2, \\ \mathfrak{g}_5 &:= \mathbf{x}_7\mathbf{x}_3 + x_2x_3 - 3x_1x_2 + 3x_1x_4 + 2x_1x_3 - 3tx_4 - 3tx_3 + 3tx_2 - \frac{3}{2}tx_1 + \frac{3}{2}t^2 \\ \mathfrak{g}_6 &:= \mathbf{x}_7\mathbf{x}_2 + x_2x_3 - 3x_1x_2 + 3x_1x_4 + 2x_1x_3 - \frac{1}{2}x_7t - 3tx_4 - \frac{5}{2}tx_3 + \frac{5}{2}tx_2 - \frac{3}{2}tx_1 + \frac{7}{4}t^2 \\ \mathfrak{g}_7 &:= \mathbf{x}_7\mathbf{x}_1 + x_1x_2 - x_1x_4 - x_1x_3 - x_7t + tx_4 + tx_3 - tx_2, \\ \mathfrak{g}_8 &:= \mathbf{x}_6^2 - 3x_2x_3 + x_1x_2 + x_1x_4 - x_1x_3 + x_3^2 + x_2^2 + x_1^2 + 2tx_6 - tx_4 + \frac{1}{2}tx_3 + 2tx_2 + \frac{1}{2}tx_1 - \frac{13}{4}t^2 \end{aligned}$$

$$\begin{aligned}
\mathfrak{g}_9 &:= \mathbf{x}_6\mathbf{x}_5 - x_1x_2 + x_1x_4 + x_1x_3 - tx_6 - tx_4 - tx_3 + tx_2 - \frac{1}{2}tx_1 + \frac{1}{2}t^2, \\
\mathfrak{g}_{10} &:= \mathbf{x}_6\mathbf{x}_4 + x_1x_2 - x_1x_4 - x_2x_3 - tx_6 + tx_4 + \frac{1}{2}tx_3 - tx_2 + \frac{1}{2}tx_1 - \frac{1}{2}t^2, \\
\mathfrak{g}_{11} &:= \mathbf{x}_6\mathbf{x}_3 + x_2x_3 - 4x_1x_2 + 4x_1x_4 + 2x_1x_3 - 4tx_4 - \frac{5}{2}tx_3 + 4tx_2 - 2tx_1 + 2t^2, \\
\mathfrak{g}_{12} &:= \mathbf{x}_6\mathbf{x}_2 + x_2x_3 - 4x_1x_2 + 4x_1x_4 + 2x_1x_3 - \frac{1}{2}tx_6 - 4tx_4 - \frac{5}{2}tx_3 + 4tx_2 - 2tx_1 + 2t^2, \\
\mathfrak{g}_{13} &:= \mathbf{x}_6\mathbf{x}_1x_1x_6 - x_1x_3 - tx_6 + tx_3, \\
\mathfrak{g}_{14} &:= \mathbf{x}_5^2 + 2x_3^2 + 2x_2^2 + 2x_1x_4 - 3x_2x_3 - x_1^2 - 14tx_5 - 2tx_4 - \frac{5}{2}tx_3 + 6tx_2 - 4tx_1 + \frac{29}{2}t^2, \\
\mathfrak{g}_{15} &:= \mathbf{x}_5\mathbf{x}_4 + x_1x_4 - tx_5 - 2tx_4 - tx_1 + 2t^2, \\
\mathfrak{g}_{16} &:= \mathbf{x}_5\mathbf{x}_3 - x_1x_2 + x_1x_4 + x_1x_3 + x_3x_5 - tx_4 - 2tx_3 + tx_2 - \frac{1}{2}tx_1 + \frac{1}{2}t^2, \\
\mathfrak{g}_{17} &:= \mathbf{x}_5\mathbf{x}_2 + x_1x_4 - \frac{1}{2}tx_5 - tx_4 - tx_2 - tx_1 + \frac{3}{2}t^2, \\
\mathfrak{g}_{18} &:= \mathbf{x}_5\mathbf{x}_1x_1x_5 - x_1x_4 - tx_5 + tx_4, \\
\mathfrak{g}_{19} &:= \mathbf{x}_4^2 + 2x_2x_3 + 11x_1x_2 - 7x_1x_4 - 8x_1x_3 - 4x_3^2 + x_1^2 - x_2^2 - 20x_7t + 37tx_4 + 15tx_3 - 14tx_2 + \frac{7}{2}tx_1 - \frac{95}{4}t^2, \\
\mathfrak{g}_{20} &:= \mathbf{x}_4\mathbf{x}_3 + x_2x_3 - 3x_1x_2 + 3x_1x_4 + 2x_1x_3 - 3tx_4 - \frac{7}{2}tx_3 + 3tx_2 - \frac{3}{2}tx_1 + \frac{3}{2}t^2, \\
\mathfrak{g}_{21} &:= \mathbf{x}_4\mathbf{x}_2 + x_2x_3 - 2x_1x_2 + 3x_1x_4 + x_1x_3 - \frac{7}{2}tx_4 - \frac{3}{2}tx_3 + tx_2 - 2tx_1 + \frac{5}{2}t^2, \\
\mathfrak{g}_{22} &:= \mathbf{x}_4\mathbf{x}_1^2 + x_1^3 - 2tx_4x_1 + 4tx_2x_1 + t^2x_4 - 4t^2x_2 - 5t^2x_1 + 4t^3, \\
\mathfrak{g}_{23} &:= \mathbf{x}_3^3 - 2tx_3^2 + 6tx_3x_2 - 4tx_4x_1 + 4tx_2x_1 + 4t^2x_4 - 3t^2x_3 - 4t^2x_2 + 2t^2x_1 - 2t^3, \\
\mathfrak{g}_{24} &:= \mathbf{x}_3^2\mathbf{x}_2 - \frac{1}{2}x_1^3 - \frac{1}{2}tx_3^2 + 4tx_3x_2 - 6tx_4x_1 - 6tx_3x_1 + 8tx_2x_1 - \frac{1}{2}tx_1^2 + 6t^2x_4 + 4t^2x_3 - 8t^2x_2 + \frac{9}{2}t^2x_1 - \frac{7}{2}t^3, \\
\mathfrak{g}_{25} &:= \mathbf{x}_3^2\mathbf{x}_1 - x_1^3 - tx_3^2 - 8tx_4x_1 - 8tx_3x_1 + 12tx_2x_1 - tx_1^2 + 8t^2x_4 + 8t^2x_3 - 12t^2x_2 + 7t^2x_1 - 5t^3, \\
\mathfrak{g}_{26} &:= \mathbf{x}_3\mathbf{x}_2^2x_2x_3 - \frac{1}{2}x_1^3 + 3tx_3x_2 - 6tx_4x_1 - 6tx_3x_1 + 8tx_2x_1 - \frac{1}{2}tx_1^2 + 6t^2x_4 + \frac{17}{4}t^2x_3 - 8t^2x_2 + \frac{9}{2}t^2x_1 - \frac{7}{2}t^3, \\
\mathfrak{g}_{27} &:= \mathbf{x}_3\mathbf{x}_2\mathbf{x}_1 - x_1^3 - tx_3x_2 - 8tx_4x_1 - \frac{17}{2}tx_3x_1 + 12tx_2x_1 - tx_1^2 + 8t^2x_4 + \frac{17}{2}t^2x_3 - 12t^2x_2 + 7t^2x_1 - 5t^3, \\
\mathfrak{g}_{28} &:= \mathbf{x}_3\mathbf{x}_1^2 - 12tx_4x_1 - 6tx_3x_1 + 12tx_2x_1 + 12t^2x_4 + 5t^2x_3 - 12t^2x_2 + 6t^2x_1 - 6t^3, \\
\mathfrak{g}_{29} &:= \mathbf{x}_2^3 + \frac{1}{2}x_1^3 + \frac{5}{2}tx_2^2 - 2tx_4x_1 - 6tx_3x_1 + 12tx_2x_1 + \frac{1}{2}tx_1^2 + 2t^2x_4 + 6t^2x_3 - \frac{61}{4}t^2x_2 - \frac{13}{2}t^2x_1 + \frac{51}{8}t^3, \\
\mathfrak{g}_{30} &:= \mathbf{x}_2^2\mathbf{x}_1 - \frac{3}{2}x_1^3 - tx_2^2 - 10tx_4x_1 - 6tx_3x_1 + 7tx_2x_1 - \frac{3}{2}tx_1^2 + 10t^2x_4 + 6t^2x_3 - 7t^2x_2 + \frac{55}{4}t^2x_1 - \frac{43}{4}t^3, \\
\mathfrak{g}_{31} &:= \mathbf{x}_2\mathbf{x}_1^2 + \frac{1}{2}x_1^3 - 10tx_4x_1 - 6tx_3x_1 + 14tx_2x_1 + 10t^2x_4 + 6t^2x_3 - 15t^2x_2 + \frac{1}{2}t^2x_1 - t^3, \\
\mathfrak{g}_{32} &:= \mathbf{x}_1^4 + 64t^2x_4x_1 - 32t^2x_2x_1 - 6t^2x_1^2 - 64t^3x_4 + 32t^3x_2 - 40t^3x_1 + 45t^4.
\end{aligned}$$

As a consequence we get the following result.

**Theorem 7.3.** *Every Gorenstein local algebra of the type  $(1, 7, 7, 1)$  is smoothable.*

*Proof.* We refer to [23] for classical and recent results about this kind of problem. Our proof consists in finding a Gorenstein point of type  $(1, 7, 7, 1)$  that belongs to the irreducible component  $\mathcal{H}_{\text{Lex}}$  and that is smooth in the Hilbert scheme. These facts imply that our point is smoothable and that all the other Gorenstein points of the same type belong to  $\mathcal{H}_{\text{Lex}}$ , i.e. are smoothable too.

We observe that the above marked set  $\mathfrak{G}_t$  turns out to be a  $[j, 3]$ -marked basis, so that it defines a family  $\mathcal{F}_1$  which is flat over  $\mathbb{A}^1$ , hence we have an embedding  $\mathbb{A}^1 \hookrightarrow \mathcal{M}f(j, 3)$ . Let us denote by  $\mathfrak{i}_\tau$  the ideal generated by the specialization  $t \mapsto \tau$  of the  $\mathfrak{G}_t$ . For every  $\tau \neq 0$ , the ideal  $\mathfrak{i}_\tau$  defines a scheme composed by 11 simple points and a multiple structure over a  $12^{\text{th}}$  point, which is Gorenstein of type  $(1, 3, 1)$ , hence smoothable because of already known results. This guarantees that also the ideal  $\mathfrak{i}_\tau$  is smoothable, that is it belongs to the component  $\mathcal{H}_{\text{Lex}}$  of  $\text{Hilb}_{16}^7$  (in particular, it belongs to  $\mathcal{M}_1$ ). For  $\tau = 0$ , the ideal  $\mathfrak{i}_0$  defines a multiple structure over a single point which is Gorenstein of type  $(1, 7, 7, 1)$  and it is smoothable too, by the flatness of the family  $\mathcal{F}_1$ . Furthermore, the dimension of the Zariski tangent space to  $\text{Hilb}_{16}^7$  at the point  $\mathfrak{i}_0$  is precisely  $112 = 16 \times 7$ . Hence  $\mathcal{H}_{\text{Lex}}$  is the only irreducible component of  $\text{Hilb}_{16}^7$  containing  $\mathfrak{i}_0$  and  $\mathfrak{i}_0$  is smooth in  $\text{Hilb}_{16}^7$ .  $\square$



Theorem 7.3 covers the case  $r' = 7$ , which is the single value not treated in the range considered by [23, Lemma 6.21].

There is a second irreducible component  $\mathcal{M}_2$  of  $\mathcal{M}f(j, 3)$  whose dimension is 161. On this component  $j$  is smooth, hence  $\mathcal{M}_2$  turns out to be isomorphic to an affine space  $\mathbb{A}^{161}$  by Theorem 7.2. We get a parametrization of  $\mathcal{M}_2$  choosing a suitable subset of 161 variables  $C_2 \subset C$  as parameters: for instance we can take  $C_2 := C \setminus \text{in}_{\prec}(\mathfrak{T})$ , where  $\mathfrak{T}$  is the ideal of the Zariski tangent space to  $\text{Hilb}_{16}^7$  at a general point of  $\mathcal{M}_2$  and the initial ideal is made w.r.t. any term order  $\prec$ . The parametrization we obtain in this way is given by polynomials. The general ideal  $\mathfrak{l}$  in  $\mathcal{M}_2$  defines a smooth point of  $\text{Hilb}_{16}^7$  and the corresponding scheme is the union of a simple point and a non reduced structure of multiplicity 15 on a different one.

For instance let us consider the ideal  $\mathfrak{l}$  defined by the following  $[j, 3]$ -marked basis.

$$\begin{aligned}
f_1 &:= \mathbf{x}_7^2 + x_3x_2 - x_3x_1 - x_2x_1 - x_1^2, \\
f_2 &:= \mathbf{x}_7\mathbf{x}_6 + x_4x_1 - x_3^2 - x_3x_2 + x_2^2 + x_2x_1 - x_1^3 + 2x_1^2, \\
f_3 &:= \mathbf{x}_7\mathbf{x}_5 - x_4x_1 - x_3x_1 - x_2^2 - x_1^3 + 4x_1^2, \\
f_4 &:= \mathbf{x}_7\mathbf{x}_4 - x_3^2 + x_2^2 + x_2x_1 + x_1^3 - 2x_1^2, \\
f_5 &:= \mathbf{x}_7\mathbf{x}_3 + x_4x_1 + x_3x_1 - x_2x_1 - x_1^3, \\
f_6 &:= \mathbf{x}_7\mathbf{x}_2 + x_3x_2 - x_3x_1 + x_2x_1 + x_1^3 - 3x_1^2, \\
f_7 &:= \mathbf{x}_7\mathbf{x}_1 - x_1^3 + x_3^2 + x_3x_2 - x_4x_1 - x_3x_1 - x_2x_1, \\
f_8 &:= \mathbf{x}_6^2 + x_3x_2 + x_3x_1 - x_2^2 - x_2x_1 + x_1^3 - 2x_1^2, \\
f_9 &:= \mathbf{x}_6\mathbf{x}_5 + x_4x_1 - x_3^2 + x_3x_2 - x_2^2 + x_2x_1 - x_1^2, \\
f_{10} &:= \mathbf{x}_6\mathbf{x}_4 + x_4x_1 + x_3^2 - x_3x_1 - x_1^3 + x_1^2, \\
f_{11} &:= \mathbf{x}_6\mathbf{x}_3 + x_4x_1 + x_3x_2 - x_2^2 + x_2x_1, \\
f_{12} &:= \mathbf{x}_6\mathbf{x}_2 - x_4x_1 - x_3^2 + x_3x_2 + x_3x_1 - x_2^2 + x_2x_1 + x_1^3 - x_1^2, \\
f_{13} &:= \mathbf{x}_6\mathbf{x}_1 + x_1^3 - x_3^2 + x_3x_2 - x_4x_1 + x_3x_1 - x_1^2, \\
f_{14} &:= \mathbf{x}_5^2 - x_4x_1 + x_3^2 + x_3x_2 - x_2^2 - x_1^3 - x_1^2, \\
f_{15} &:= \mathbf{x}_5\mathbf{x}_4 - x_4x_1 + x_3x_2 + x_3x_1 - x_1^3 - x_1^2, \\
f_{16} &:= \mathbf{x}_5\mathbf{x}_3 + x_4x_1 + x_3^2 - x_3x_2 - x_3x_1 - x_2^2 + x_1^3 + 2x_1^2, \\
f_{17} &:= \mathbf{x}_5\mathbf{x}_2 + x_4x_1 - x_3^2 - x_3x_2 + x_2^2 - x_2x_1 - x_1^3 + 4x_1^2, \\
f_{18} &:= \mathbf{x}_5\mathbf{x}_1 + x_1^3 + x_3^2 - x_4x_1 - x_3x_1 - x_2x_1 + x_1^2, \\
f_{19} &:= \mathbf{x}_4^2 + x_3^2 + x_3x_1 - x_2^2 - x_2x_1 + x_1^3 - x_1^2, \\
f_{20} &:= \mathbf{x}_4\mathbf{x}_3 + x_3x_1 + x_2^2 + x_2x_1 + x_1^3 - 4x_1^2, \\
f_{21} &:= \mathbf{x}_4\mathbf{x}_2 + x_4x_1 - x_3x_2 + x_3x_1 + x_2^2 + x_2x_1 - x_1^3 - x_1^2, \\
f_{22} &:= \mathbf{x}_4\mathbf{x}_1^2, \quad f_{23} := \mathbf{x}_3^3 - x_1^3, \quad f_{24} := \mathbf{x}_3^2\mathbf{x}_2 - x_1^3, \quad f_{25} := \mathbf{x}_3^2\mathbf{x}_1 - x_1^3, \\
f_{26} &:= \mathbf{x}_3\mathbf{x}_2^2 - x_1^3, \quad f_{27} := \mathbf{x}_3\mathbf{x}_2\mathbf{x}_1 - x_1^3, \quad f_{28} := \mathbf{x}_3\mathbf{x}_1^2 - x_1^3, \\
f_{29} &:= \mathbf{x}_2^3 - x_1^3, \quad f_{30} := \mathbf{x}_2^2\mathbf{x}_1 - x_1^3, \quad f_{31} := \mathbf{x}_2\mathbf{x}_1^2 - x_1^3, \quad f_{32} := \mathbf{x}_1^4 - x_1^3.
\end{aligned}$$

We can verify that  $\mathfrak{l}$  is smooth on  $\text{Hilb}_{16}^7$  because the dimension of the Zariski tangent space to  $\text{Hilb}_{16}^7$  at this point is 161.

Finally we can find a third irreducible component  $\mathcal{M}_3$  of  $\mathcal{M}f(j, 3)$  in the following way. If we replace by 0 the variables in a suitable subset  $C_3 \subset C$ , we obtain a family  $\mathcal{F}_3$  which is flat over  $\mathbb{A}^{116}$ . A general point  $\mathfrak{v}$  of  $\mathcal{F}_3$ , for instance the one defined by the ideal generated by the following  $[j, 3]$ -marked basis, is a non-reduced structure over a point.

$$\begin{aligned}
h_1 &:= \mathbf{x}_7^2 - x_3^2 + x_3x_2 - 2x_4x_1 + 2x_3x_1, \quad h_2 := \mathbf{x}_7\mathbf{x}_6 + 3x_3^2 + 4x_3x_2 - 4x_2^2 - x_4x_1 + 3x_3x_1, \\
h_3 &:= \mathbf{x}_7\mathbf{x}_5 - 2x_3^2 + x_3x_2 + x_2^2 + 4x_4x_1 + 3x_3x_1, \quad h_4 := \mathbf{x}_7\mathbf{x}_4 - 2x_3^2 + 3x_3x_2 + x_2^2 - 3x_4x_1 + x_3x_1, \\
h_5 &:= \mathbf{x}_7\mathbf{x}_3 + 2x_3^2 + x_3x_2 - x_2^2 - x_4x_1 - 4x_3x_1, \quad h_6 := \mathbf{x}_7\mathbf{x}_2 - x_3^2 + x_3x_2 - 4x_2^2 + x_4x_1 + 4x_3x_1, \\
h_7 &:= \mathbf{x}_7\mathbf{x}_1 - 2x_3^2 - x_3x_2 + 2x_2^2 - 3x_4x_1 + x_3x_1, \quad h_8 := \mathbf{x}_6^2 - x_3x_2 + 3x_2^2 + 2x_4x_1, \\
h_9 &:= \mathbf{x}_6\mathbf{x}_5 + 2x_3^2 + 2x_2^2 - 4x_4x_1 - 3x_3x_1, \quad h_{10} := \mathbf{x}_6\mathbf{x}_4 + x_3^2 + x_3x_2 - x_4x_1 + 2x_3x_1, \\
h_{11} &:= \mathbf{x}_6\mathbf{x}_3 - 4x_3^2 + 3x_2^2 - 2x_3x_1, \quad h_{12} := \mathbf{x}_6\mathbf{x}_2 - 2x_3^2 + 3x_2^2 + x_4x_1 + 3x_3x_1, \\
h_{13} &:= \mathbf{x}_6\mathbf{x}_1 - 2x_3^2 + 2x_3x_2 - 4x_2^2 - x_4x_1 - 4x_3x_1, \quad h_{14} := \mathbf{x}_5^2 - 4x_3^2 - 2x_3x_2 - 2x_2^2,
\end{aligned}$$

$$\begin{aligned}
h_{15} &:= \mathbf{x}_5 \mathbf{x}_4 - 2x_3 x_2 - 4x_2^2 + 4x_4 x_1 - x_3 x_1, & h_{16} &:= \mathbf{x}_5 \mathbf{x}_3 - 4x_2^2 - x_4 x_1 + x_3 x_1, \\
h_{17} &:= \mathbf{x}_5 \mathbf{x}_2 + x_3^2 + 3x_3 x_2 + 3x_2^2 - x_4 x_1 - 4x_3 x_1, & h_{18} &:= \mathbf{x}_5 \mathbf{x}_1 - 3x_3^2 - 2x_2^2 - 3x_4 x_1 + x_2 x_1, \\
h_{19} &:= \mathbf{x}_4^2 - 2x_3^2 - 2x_3 x_2 + 4x_2^2 + x_4 x_1 + x_3 x_1, & h_{20} &:= \mathbf{x}_4 \mathbf{x}_3 + x_3^2 + x_3 x_2 + 4x_2^2 - 2x_4 x_1 - 4x_3 x_1, \\
h_{21} &:= \mathbf{x}_4 \mathbf{x}_2 + x_3^2 + 2x_3 x_2 + 4x_2^2 - 4x_4 x_1 - 2x_3 x_1, & h_{22} &:= \mathbf{x}_4 \mathbf{x}_1^2, & h_{23} &:= \mathbf{x}_3^3, \\
h_{24} &:= \mathbf{x}_3^2 \mathbf{x}_2, & h_{25} &:= \mathbf{x}_3^2 \mathbf{x}_1, & h_{26} &:= \mathbf{x}_3 \mathbf{x}_2^2, & h_{27} &:= \mathbf{x}_3 \mathbf{x}_2 \mathbf{x}_1, & h_{28} &:= \mathbf{x}_3 \mathbf{x}_1^2, & h_{29} &:= \mathbf{x}_2^3, \\
h_{30} &:= \mathbf{x}_2^2 \mathbf{x}_1, & h_{31} &:= \mathbf{x}_2 \mathbf{x}_1^2 + 3x_1^3, & h_{32} &:= \mathbf{x}_1^4.
\end{aligned}$$

Since  $\mathcal{F}_3$  is a flat family over  $\mathbb{A}^{116}$ , we have an embedding  $\mathbb{A}^{116} \hookrightarrow \mathcal{M}_3$ . Hence the irreducible component  $\mathcal{M}_3$  of  $\mathcal{M}(\mathbf{j}, 3)$  cannot coincide with  $\mathcal{M}_1$  because  $\dim(\mathcal{M}_3) \geq 116 > 112 = \dim(\mathcal{M}_1)$ . On the other hand  $\mathcal{M}_3$  cannot coincide with  $\mathcal{M}_2$  because the dimension of the Zariski tangent space to  $\mathcal{Hilb}_{16}^7$  at a general point of  $\mathcal{F}_3$ , as for instance  $\mathbf{v}$ , is 153, while at every point of  $\mathcal{M}_2$  such dimension is  $\geq 161$ .

Therefore there are at least three irreducible components of  $\mathcal{Hilb}_{16}^7$  passing through  $\mathbf{j}$ .

**7.2. Smoothability of local Gorenstein algebras  $(1, 5, 5, 1)$ .** We consider  $\mathcal{Hilb}_{12}^5$  that parameterizes 0-dimensional subschemes of  $\mathbb{P}^5$  of length 12. As in the previous case, we identify every point of  $\mathcal{Hilb}_{12}^5$  with an ideal in  $R = K[x_1, \dots, x_5]$ , non-necessarily homogeneous, with affine Hilbert polynomial  $p(t) = 12$  and we order the variables as  $x_5 > \dots > x_1$ .

The ideal  $\mathbf{i}_0$  of Example 5.9 corresponds to a point of  $\mathcal{Hilb}_{12}^5$  and the quotient  $R/\mathbf{i}_0$  is a local Gorenstein algebra with Hilbert Function  $(1, 5, 5, 1)$ . Its initial ideal w.r.t. the term order **Lex** is the ideal  $\mathbf{j}$  presented therein. We observe that  $\mathbf{j}$  is strongly stable and affine 3-segment with weight vector  $\omega = [8, 7, 5, 4, 3]$ .

We can construct the  $[\mathbf{j}, 3]$ -marked set  $\mathfrak{G}$  in  $K[C, x_1, \dots, x_5]$  (as in (5.1)) and apply the Algorithm MARKEDSCHEME with input  $\mathbf{j}$  and 3. We get a set of 576 polynomials in  $K[C]$ , where  $|C| = 204$ , that generate the ideal  $\mathfrak{A}$  of  $\mathcal{M}(\mathbf{j}, 3)$ . Using  $\mathfrak{A}$  we compute the Zariski tangent space to  $\mathcal{M}(\mathbf{j}, 3)$  at the point corresponding to  $\mathbf{i}_0$ . The dimension of this tangent space is  $60 = 12 \times 5$ , the one expected if  $\mathbf{i}_0$  were a smooth point of the **Lex**-component of  $\mathcal{Hilb}_{12}^5$ . If this is the case, the computation of the tangent space also highlights a special set  $\tilde{C}$  of 60 variables  $C$  that can give a local parametrization around  $\mathbf{i}_0$ . Specializing in  $\mathfrak{A}$  all the variables in  $\tilde{C}$  to 0 except a suitable one which we choose as a parameter  $T$ , we can determine a specialization for every variable in  $C$  depending on  $T$ . In this way we obtain the following set of polynomials, marked over  $\mathbf{j}$ :

$$\begin{aligned}
f_1 &:= \mathbf{x}_5^2 + 4x_1^2 + \frac{17}{3}x_1x_2 - \frac{83}{12}x_1x_3 - \frac{23}{4}x_2x_3, \\
f_2 &:= \mathbf{x}_4\mathbf{x}_5 - \frac{3}{4}x_2x_3 - \frac{5}{4}x_1x_3 + x_1x_2, \\
f_3 &:= \mathbf{x}_4^2 - Tx_4 + x_2T + \frac{25}{6}x_2x_3 + x_2^2 + \frac{71}{18}x_1x_3 - \frac{28}{9}x_1x_2 - 5x_1^2, \\
f_4 &:= \mathbf{x}_3\mathbf{x}_5 - \frac{3}{4}x_2x_3 + \frac{3}{4}x_1x_3 - x_1x_2, \\
f_5 &:= \mathbf{x}_3\mathbf{x}_4 - x_2x_3, \\
f_6 &:= \mathbf{x}_3^2 - \frac{85}{24}x_2x_3 - \frac{317}{72}x_1x_3 + \frac{71}{18}x_1x_2 + 2x_1^2, \\
f_7 &:= \mathbf{x}_2\mathbf{x}_5 - \frac{3}{4}x_2x_3 - \frac{5}{4}x_1x_3 + x_1x_2, \\
f_8 &:= \mathbf{x}_2\mathbf{x}_4 - x_2x_3 - x_1x_3 + x_1x_2, \\
f_9 &:= \mathbf{x}_1\mathbf{x}_5 - \frac{1}{4}x_2x_3 + \frac{1}{4}x_1x_3 - x_1x_2, \\
f_{10} &:= \mathbf{x}_1\mathbf{x}_4 - x_1x_2, \\
f_{11} &:= \mathbf{x}_2^2\mathbf{x}_3 + x_1^3, \\
f_{12} &:= \mathbf{x}_2^3 - x_2x_3T - x_3x_1T + Tx_2^2 + x_2x_1T + \frac{5}{9}x_1^3, \\
f_{13} &:= \mathbf{x}_2\mathbf{x}_1\mathbf{x}_3 - \frac{11}{9}x_1^3, & f_{14} &:= \mathbf{x}_1\mathbf{x}_2^2 - \frac{8}{9}x_1^3, & f_{15} &:= \mathbf{x}_1^2\mathbf{x}_3 + x_1^3, & f_{16} &:= \mathbf{x}_1^2\mathbf{x}_2 + \frac{2}{3}x_1^3, & f_{17} &:= \mathbf{x}_1^4.
\end{aligned}$$

This marked set is, by construction, a  $[\mathbf{j}, 3]$ -marked basis, so that it defines a family  $\mathcal{F}_1$  (different from that of Example 5.9) which is flat over  $\mathbb{A}^1$ , hence we have an embedding  $\mathbb{A}^1 \hookrightarrow \mathcal{M}(\mathbf{j}, 3)$ . Specializing  $T$  to 0 we obtain a set of generators for  $\mathbf{i}_0$ , while specializing to a non zero value the ideal we obtain defines a scheme composed by 2 simple points and a multiple structure over a  $3^{th}$  point, which is Gorenstein of type  $(1, 4, 4, 1)$  and then smoothable by [8].

As a consequence, by the same reasoning applied in the proof of Theorem 7.3, we get the following result.

**Theorem 7.4.** *Every Gorenstein local algebra of the type  $(1, 5, 5, 1)$  is smoothable.*

This last result has been independently obtained by J. Jelisiejew in [24] by different tools.

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